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Farlie-Gumbel-Morgenstern copula-based modeling of non-conforming rate in bivariate lifetime data

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Abstract: The lifetime performance index is a widely accepted measure for evaluating a process capability in terms of both its actual and expected performance. In manufacturing contexts, a product is typically considered acceptable if its lifetime exceeds a specified minimum threshold. This study investigates statistical inference for the nonconforming rate of a two-component system, in which the dependence between the two lifetime characteristics is modeled using the Farlie-Gumbel-Morgenstern copula. The marginal distributions of both components are assumed to follow Weibull distributions. A numerical example is presented to illustrate the practical application of the proposed approaches.

Keywords: Bivariate Weibull distribution; Farlie-Gumbel-Morgenstern copula; Lifetime performance index; Monte Carlo procedure; Non-conforming rate.
Mathematics Subject Classification (2010): 93D25, 93E03, 93E20.

1 Introduction

To ensure that a process aligns with the performance requirements of engineers or clients, capability indices are employed as quantitative tools. Montgomery (1985) proposed a specific index, denoted as C_L, to assess quality characteristics where higher values are more desirable-often referred to as the “larger-the-better” scenario:

$$C_L = \frac{\mu - L}{\sigma},$$

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where μ and σ denote the mean and standard deviation of the process, respectively, and L represents the minimum acceptable lifetime of the product. In a related development, Kane (1986) introduced the C_{PL} index to assess the lifetime performance of electronic components.

A complete sample occurs when the lifetimes of all observed items are fully recorded. Prior research has investigated the estimation of C_L using complete datasets from classical lifetime distributions. As an illustration, Tong et al. (2002) developed the UMVUE for C_L and proposed a testing framework assuming that the data follow a one-parameter exponential distribution. Moreover, the lifetime performance index has been utilized to assess product durability using censored samples and record-value data. However, to date, no study has specifically addressed situations in which product lifetime is simultaneously influenced by two distinct characteristics.

It is worth noting that, while the lifetime performance index C_L does not exceed 1 for an exponential distribution, this is not the case for a Weibull distribution. For a Weibull model, C_L can take any real value (including values greater than 1) depending on the shape and scale parameters. Consequently, there is no fixed upper bound, and a process is considered more capable as C_L increases.

If the lifetime of a product, denoted by X, is greater than the minimum required value L, the product is considered conforming. Conversely, products whose lifetimes fall short of this benchmark are regarded as nonconforming. Because longer lifetimes are associated with improved quality, the nonconformity rate is calculated as

$$P_L = P(X < L) = \int_0^L f(x)dx. \tag{1}$$

In actuarial science, scenarios involving the simultaneous risk of failure for two lives-such as joint life insurance or annuity contracts-require analysis based on the joint distribution of their lifetimes. In this study, we consider a setting where product lifetime is influenced by two interdependent characteristics, modeled through the Farlie-Gumbel-Morgenstern (FGM) copula. We then proceed to derive statistical inferences for the nonconformance probability P_L. Abbasi Ganji et al. (2022) have conducted a related investigation using a different distributional approach.

The remainder of this paper is organized as follows. Section 2 provides essential background and theoretical preliminaries. In Section 3, the non-conforming rate is derived under the FGM copula framework with Weibull marginals. In this Section, we address the point and interval estimation of this rate using the maximum likelihood and asymptotic approaches. Also, This Section discusses hypothesis testing procedures and presents a Monte Carlo simulation strategy to estimate the associated P-value. Section 4 includes a numerical example to demonstrate the practicality of the method, and Section 5 concludes the study. Additional derivations are detailed in the Appendix.

2 Preliminary

This section provides the necessary theoretical background. Given its well-established prominence and flexibility in reliability engineering, the Weibull distribution is selected to model the marginal lifetimes of the components (Lawless, 2011; Rinne, 2008). The key advantage of the Weibull distribution lies in its shape parameter β, which enables the modeling of decreasing ($\beta < 1$), constant ($\beta = 1$, reducing to the exponential distribution), or increasing ($\beta > 1$) failure rates. This versatility makes it suitable for representing a wide range of product aging and wear-out phenomena commonly encountered in engineering applications (Abernethy, 2006). Furthermore, the Weibull distribution offers a relatively tractable mathematical form for both its probability density function (pdf) and cumulative distribution function (cdf), which facilitates the derivation of likelihood functions and subsequent statistical inferences (Dodson, 2006). While the methodological framework developed in this study is presented using Weibull marginals for clarity and to align with prevalent practice, the underlying copula-based approach is conceptually generalizable to other common lifetime distributions (e.g., log-normal, gamma) should the application context warrant it.

In this section, we focus on the Weibull distribution, a widely and commonly adopted model in lifetime data analysis. Its popularity in fields such as reliability engineering stems from its adaptable shape and analytical simplicity.

The pdf of the two-parameter Weibull distribution, which is widely used in reliability analysis, can be expressed as

$$f_X(x,\beta,\eta) = \frac{\beta}{\eta}(\frac{x}{\eta})^{\beta-1}e^{-(\frac{x}{\eta})^{\beta}}, \quad x>0, \quad \beta,\eta>0.$$

and its cdf is as

$$F_X(x,\beta,\eta) = 1 - e^{-(\frac{x}{\eta})^{\beta}}, \quad x>0, \quad \beta,\eta>0.$$

Let $X \sim W(\beta,\ \eta)$, where β and η represent the shape and scale parameters, respectively. The expected value and standard deviation of X,respectively, are

$$\mu = \eta\Gamma(\frac{1}{\beta}+1), \quad \sigma = \eta\sqrt{\Gamma(\frac{2}{\beta}+1) - \Gamma(\frac{1}{\beta}+1)},$$

where $\Gamma(\cdot)$ is the gamma function. So, the lifetime performance index, C_L, is given by

$$C_L = \frac{\eta\Gamma(\frac{1}{\beta}+1)}{\eta\sqrt{\Gamma(\frac{2}{\beta}+1) - \Gamma(\frac{1}{\beta}+1)}}.$$

2.1 Copula

The concept of copula, a fundamental construct in statistical theory, facilitates the combination of marginal distributions into a joint multivariate distribution. Introduced by Sklar (1959), copulas allow for the explicit modeling of dependence structures that are otherwise not captured by the marginals alone. Sklar's theorem, widely regarded as the foundational result in copula theory, plays a central role in nearly all practical implementations involving copulas.

Theorem 2.1 (Sklar's Therorem)**.** *Let H be an n-dimensional distribution function with margins $F_1, F_2, \ldots, F_n$. Then, there exists an n-copula C such that for all $\boldsymbol{x}$ in $\bar{R}^n$,*

$$H(x_1, x_2, \ldots, x_n) = C\big(F_1(x_1), F_2(x_2), \ldots, F_n(x_n)\big). \tag{2}$$

When the marginal distributions $F_1, F_2, \ldots, F_n$ are continuous, the corresponding copula C is uniquely determined. In cases where the marginals are not fully continuous, uniqueness holds on the image of each F_i, denoted by $RanF_1 \times RanF_2 \times \ldots \times RanF_n$. Conversely, if a function C is an n-dimensional copula and $F_1, F_2, \ldots, F_n$ are cdfs, then the function H defined as in (2) is a valid multivariate distribution with margins $F_1, F_2, \ldots, F_n$.

Sklar's theorem implies that any joint distribution involving n random variables can be represented by a copula function coupled with the marginal distributions. This decomposition isolates the dependence structure, thus enabling separate and accurate modeling of each marginal component.

A widely recognized class of copulas is the FGM family, originally introduced by Morgenstern (1956), Farlie (1960), and Gumbel (1960). The FGM copula is expressed as

$$C(u, v) = uv\big[1 + \theta(1-u)(1-v)\big], \quad u, v \in [0, 1],$$

where $\theta \in [-1, 1]$ acts as the dependence parameter. When $\theta = 0$, the copula reduces to the independence case. The variables u and v are assumed to follow uniform distributions on the interval $(0, 1)$.

$$c(u, v) = \big[1 + \theta(2u-1)(2v-1)\big], \quad u, v \in [0, 1].$$

Although, the FGM copula is analytically tractable and conceptually simple, its practical use is restricted by a limited ability to capture strong dependence structures. Specifically, the attainable correlation between variables under this model lies within a narrow range, approximately from $\rho \in [-\frac{1}{3}, \frac{1}{3}]$, which may be insufficient for modeling highly correlated data. Moreover, the FGM copula does not exhibit tail dependence, meaning it cannot model situations where extreme values in one variable are likely to coincide with extreme values in another. This makes it particularly suitable for modeling weak-to-moderate dependence in settings where simultaneous extreme events are not a primary concern, but unsuitable for data exhibiting strong correlation or tail-dependent behavior.

For a broader treatment of copula theory and applications, readers may refer to foundational and advanced texts such as Sklar (1959), Genest and MacKay (1986), Joe (1997), Nelsen (1999), Yan (2006), Genest and Favre (2007),Wang et al. (2016), Wiboonpongse et al. (2015), Yang et al. (2015) and Zhang et al. (2016).

In recent years, there has been significant progress in the construction and application of copula models tailored for bivariate lifetime data. Susam (2022) introduced a flexible framework for generating multi-parameter generalized FGM copulas based on a customized generator function $\phi(t)$, which offers direct interpretability through Kendall's tau and enhances empirical fitting. Building on this, a subsequent study by the same author proposed a two-parameter extension of the traditional FGM copula using a compound formulation, which improves its adaptability to moderate dependence structures commonly observed in real-world datasets (Susam, 2022). On a related front, Briseno-Sanchez et al. (2024) developed a distributional copula regression model for censored bivariate survival data, embedding multivariate accelerated failure time (AFT) structures within a copula-based framework, and incorporating data-driven variable selection via gradient boosting. These advancements collectively highlight the increasing role of flexible copula-based approaches in modeling dependence structures under complex survival settings.

The primary rationale for employing the FGM copula in this methodological study is its balance between flexibility and analytical simplicity. Its closed-form expression facilitates the derivation of the joint likelihood function, the non-conformance probability P_L, and the subsequent asymptotic inference procedures (e.g., the delta method for confidence intervals). This tractability allows us to establish a clear foundational framework for capability assessment in bivariate lifetime models. We emphasize that the focus here is on presenting a coherent inference methodology for a copula-based bivariate model; the framework is conceptually extensible to other copula families, though with increased analytical complexity.

2.2 Dependence measure

According to Genest and Favre (2007), the analysis of dependence structures should rely on rank-based approaches rather than raw data values. A widely accepted, scale-invariant statistic for measuring dependence is Kendall's τ, which captures the ordinal association between two variables without assuming any specific parametric form. Let (X, Y) be a pair of random variables with joint distribution H and associated copula function C. Then,

$$\tau_{X,Y} = \tau_C = 4 \int\int_{I^2} C(u,v)dC(u,v) - 1.$$

Within the FGM copula framework, there exists a linear relationship between the dependence parameter θ and Kendall's rank correlation coefficient τ, given by $\tau_{X,Y} = 2\theta/9$. As a result, the feasible values of τ for this copula are confined to the interval $[-2/9, 2/9]$. When an empirical estimate of Kendall's τ is available from data, one can obtain an estimate for θ using the inverse relation: $\widehat{\theta} = 9\tau/2$.

One of the notable limitations of the FGM copula model is its inability to capture tail dependence between variables. In practical terms, consider two dependent lifetimes, X and Y, within a reliability framework. Tail dependence refers to the tendency of one variable to take on extreme values, given that the other variable has also exhibited an extreme outcome. In the case of the FGM family, such dependence in the tails is absent, making it unsuitable for modeling scenarios where simultaneous extreme events are likely, such as in systems prone to cascading failures or common-cause faults. Therefore, the choice of the FGM copula in this study is motivated by its mathematical tractability and its appropriateness for modeling systems where the interdependence between component lifetimes is expected to be weak or moderate, rather than strong or tail-dependent. For a more comprehensive treatment of tail dependence concepts, see Nelsen (1999).

2.3 Graphical tools for detecting dependence

To explore the nature of dependence between variables, a useful approach is to generate a scatter plot of the normalized ranks, defined as $(R_i/(n+1), S_i/(n+1))$, where R_i and S_i denote the rank positions of the observed data. This empirical plot is then visually compared with a synthetic data set simulated from the target copula model. Additionally, diagnostic tools such as the chi-plot and Kendall plot are utilized to further assess the underlying dependence structure. These graphical techniques provide intuitive insights into the type and strength of dependence between the variables.

The chi-plot, originally proposed by Fisher and Switzer (1985), Fisher and Switzer (2001), serves as a graphical diagnostic for identifying local dependence between variables. Drawing its conceptual roots from control chart methodology, the chi-plot leverages the chi-square framework to highlight deviations from independence within a bivariate data set. Importantly, this plot is constructed solely based on the rank values of the observed data, making it a nonparametric and distribution-free tool for dependence detection.

Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be an independent and identically distributed sample drawn from a bivariate distribution with joint cdf $H(X, Y)$. For each observation (X_i, Y_i), we define the following quantities to facilitate the graphical analysis of dependence structure.

$$\begin{aligned}
H_i &= \frac{1}{n-1} \sharp\{j \neq i : X_j \leq X_i, Y_j \leq Y_i\}, \\
F_i &= \frac{1}{n-1} \sharp\{j \neq i : X_j \leq X_i\}, \\
G_i &= \frac{1}{n-1} \sharp\{j \neq i : Y_j \leq Y_i\}, \\
S_i &= \text{sign}\big((F_i - \frac{1}{2})(G_i - \frac{1}{2})\big),
\end{aligned}$$

where the symbol $\sharp$ represents the count of elements satisfying the specified condition, and sign$(\cdot)$ is the sign function. Then,

$$\begin{aligned}
\lambda_i &= 4S_i \max\{(F_i - \frac{1}{2})^2, (G_i - \frac{1}{2})^2\}, \\
\chi_i &= \frac{H_i - F_i G_i}{\sqrt{F_i(1-F_i)G_i(1-G_i)}}.
\end{aligned}$$

The chi-plot is constructed by plotting each pair (λ_i, χ_i), where χ_i reflects a measure of local dependence. Under the assumption of independence, the expected value of χ_i is zero, implying that $P(X \leq u, Y \leq v) = P(X \leq u)P(Y \leq v)$. To aid in interpretation, Fisher and Switzer proposed including threshold lines at $\pm c_p/\sqrt{n}$, where c_p corresponds to a quantile such that approximately $100p\%$ of the points are expected to fall within these boundaries. The suggested values for c_p are 1.54, 1.78, and 2.18 for $p = 0.9, 0.95$, and 0.99, respectively.

To minimize the influence of potential outliers, it is recommended to include only those data pairs in the chi-plot for which $\mid \lambda_i \mid < 4(1/(n-1) - 1/2)^2$. This constraint helps maintain the reliability of the visual analysis.

The Kendall plot (commonly referred to as the K-plot), introduced by Genest and Boies (2003), is an adaptation of the probability plot concept for identifying dependence structures. This plot is constructed by displaying the points $(W_{i:n}, H_{(i)})$, where $H_{(i)}$ denotes the ordered values of the $H_{(i)}$ statistics (originally defined in the context of the chi-plot). The component $W_{i:n}$ represents the expected value of the ith order statistic from a sample drawn from the copula distribution $W = H(X,Y)$, assuming independence between the variables. Its computation is based on the following expression:

$$\begin{aligned} W_{i:n} &= n\binom{n-1}{i-1}\int_0^1 wk_0(w)\big(K_0(w)\big)^{i-1}\big(1-K_0(w)\big)^{n-i}dw, \\ K_0(w) &= w - w\log(w). \end{aligned}$$

The interpretation of the Kendall plot closely resembles that of a QQ-plot. When the plotted points lie near the 45-degree reference line, it suggests that variables X and Y are likely to be independent. Deviations from this diagonal line signal the presence of dependence: points falling predominantly above the diagonal indicate positive dependence, while points below imply negative dependence. Moreover, greater divergence from the diagonal is indicative of stronger dependence. For a more detailed discussion on the K-plot, readers may consult Genest and Boies (2003).

2.4 Assessing model appropriateness

Goodness-of-fit tests are employed to assess whether a given statistical model appropriately captures the underlying structure of a dataset. These tests essentially quantify the discrepancy between the observed data and the values predicted by the proposed model. In multivariate analysis, the Cramér–von Mises statistic is frequently utilized to examine whether a selected copula family accurately represents the dependence structure of a joint distribution. In this context, the null hypothesis $H_0 : C \in C_0$ is tested, where C_0 denotes the set of candidate copulas considered to reflect the true dependence structure.

Mathematically, the Cramér-von Mises statistic, which quantifies the deviation between the empirical and theoretical distributions, can be expressed as

$$S_{n\xi} = n\int_\xi^1 \big(C_n(w) - C_{\theta_n}(w)\big)^2 dw,$$

where, C_n represents the empirical copula derived from the observed sample (X_1, Y_1), $\ldots, (X_n, Y_n)$, while C_{θ_n} denotes the estimated copula under the null hypothesis. The parameter $\xi \in (0,1)$ serves as an arbitrary threshold.

Several researchers have explored the Cramér-von Mises test in depth. Detailed discussions can be found in studies by Genest et al. (2006), Genest and Rémillard (2008), Genest and Rivest (1993), as well as Wang and Wells (2000).

3 Non-conforming rate

Suppose the joint cdf of two Weibull variables $X_1 \sim W(\beta_1,\ \eta_1)$ and $X_2 \sim W(\beta_2,\ \eta_2)$ is based on FGM model as what follows

$$F_{X_1,X_2}(x_1,x_2,\beta_1,\beta_2,\eta_1,\eta_2,\theta) = \big(1-e^{-(\frac{x_1}{\eta_1})^{\beta_1}}\big)\big(1-e^{-(\frac{x_2}{\eta_2})^{\beta_2}}\big)\Big[1+\theta e^{-(\frac{x_1}{\eta_1})^{\beta_1}-(\frac{x_2}{\eta_2})^{\beta_2}}\Big],$$
$$x_1, x_2 > 0, \quad \beta_1,\ \beta_2,\ \eta_1,\ \eta_2 > 0. \tag{3}$$

In (1), the non-conforming rate is defined for a single component. However, in a two-component system, a product is considered *conforming* only if the lifetime of *both* characteristics exceeds their respective lower specification limits, i.e., $X_1 > L_1$ and $X_2 > L_2$. Therefore, the product is *non-conforming* if *at least one* of the components fails to meet its requirement. Hence, the system non-conformance probability is given by the union probability $P_L = P(\{X_1 \leq L_1\} \cup \{X_2 \leq L_2\})$. Applying the general addition rule of probability leads directly to the form expressed in the following

$$\begin{aligned} P_L &= P(\{X_1 \leq L_1\} \cup \{X_2 \leq L_2\}) \\ &= P(X_1 \leq L_1) + P(X_2 \leq L_2) - P(X_1 \leq L_1, X_2 \leq L_2), \end{aligned}$$

where, the joint probability $P(X_1 \leq L_1, X_2 \leq L_2)$ is captured by the joint cdf in (3), which incorporates the dependence structure via the FGM copula. Then, the non-conforming rate is obtained as

$$\begin{aligned} P_L =& F_{X_1}(L_1) + F_{X_2}(L_2) - F_{X_1,X_2}(L_1, L_2) \\ =& 2 - e^{-(\frac{L_1}{\eta_1})^{\beta_1}} - e^{-(\frac{L_2}{\eta_2})^{\beta_2}} - \left(1 - e^{-(\frac{L_1}{\eta_1})^{\beta_1}}\right)\left(1 - e^{-(\frac{L_2}{\eta_2})^{\beta_2}}\right)\left[1 + \theta e^{-(\frac{L_1}{\eta_1})^{\beta_1} - (\frac{L_2}{\eta_2})^{\beta_2}}\right]. \end{aligned} \tag{4}$$

Define y_1 and y_2 as $y_1 = e^{-(\frac{L_1}{\eta_1})^{\beta_1}}$ and $y_2 = e^{-(\frac{L_2}{\eta_2})^{\beta_2}}$, respectively. Substituting these into the previous expression yields the following simplified form:

$$P_L = 2 - y_1 - y_2 - (1 - y_1)(1 - y_2) - \theta y_1 y_2 (1 - y_1)(1 - y_2).$$

3.1 Maximum likelihood estimatiom of P_L

Given a sample of size n, denoted by $\underline{x} = (x_1, \ldots, x_n)'$, where each observation $x_1 = (x_{1i}, x_{2i})$ for $i = 1, \ldots, n$, the maximum likelihood estimator (MLE) of P_L is derived as follows

$$\widehat{P}_L = 2 - e^{-(\frac{L_1}{\widehat{\eta}_1})^{\widehat{\beta}_1}} - e^{-(\frac{L_2}{\widehat{\eta}_2})^{\widehat{\beta}_2}} - \left(1 - e^{-(\frac{L_1}{\widehat{\eta}_1})^{\widehat{\beta}_1}}\right)\left(1 - e^{-(\frac{L_2}{\widehat{\eta}_2})^{\widehat{\beta}_2}}\right)\left[1 + \theta e^{-(\frac{L_1}{\widehat{\eta}_1})^{\widehat{\beta}_1} - (\frac{L_2}{\widehat{\eta}_2})^{\widehat{\beta}_2}}\right], \tag{5}$$

where $\widehat{\beta}_1$, $\widehat{\beta}_2$, $\widehat{\eta}_1$, and $\widehat{\eta}_2$ are obtained by solving the following system of non-linear equations:

$$\begin{aligned} &\frac{n}{\beta_1} + \sum_{i=1}^{n} \log x_{1i} - n \log \eta_1 - \sum_{i=1}^{n} (\frac{x_{1i}}{\eta_1})^{\beta_1} \log(\frac{x_{1i}}{\eta_1}) \\ &\quad - \sum_{i=1}^{n} \frac{2\theta(\frac{x_{1i}}{\eta_1})^{\beta_1} \log(\frac{x_{1i}}{\eta_1}) e^{-(x_{1i}/\eta_1)^{\beta_1}} \left(2e^{-(x_{2i}/\eta_2)^{\beta_2}} - 1\right)}{1 + \theta\left(2e^{-(x_{1i}/\eta_1)^{\beta_1}} - 1\right)\left(2e^{-(x_{2i}/\eta_2)^{\beta_2}} - 1\right)} = 0, \\ &\frac{n}{\beta_2} + \sum_{i=1}^{n} \log x_{2i} - n \log \eta_2 - \sum_{i=1}^{n} (\frac{x_{2i}}{\eta_2})^{\beta_2} \log(\frac{x_{2i}}{\eta_2}) \\ &\quad - \sum_{i=1}^{n} \frac{2\theta(\frac{x_{2i}}{\eta_2})^{\beta_2} \log(\frac{x_{2i}}{\eta_2}) e^{-(x_{2i}/\eta_2)^{\beta_2}} \left(2e^{-(x_{1i}/\eta_1)^{\beta_1}} - 1\right)}{1 + \theta\left(2e^{-(x_{1i}/\eta_1)^{\beta_1}} - 1\right)\left(2e^{-(x_{2i}/\eta_2)^{\beta_2}} - 1\right)} = 0, \end{aligned}$$

$$\frac{\beta_1(\sum_{i=1}^n x_{1i}^{\beta_1})}{\eta_1^{\beta_1+1}} - \frac{n\beta_1}{\eta_1} + \sum_{i=1}^n \frac{2\theta\beta_1 \frac{x_{1i}^{\beta_1}}{\eta_1^{\beta_1+1}} e^{-(x_{1i}/\eta_1)^{\beta_1}} \left(2e^{-(x_{2i}/\eta_2)^{\beta_2}} - 1\right)}{1 + \theta\left(2e^{-(x_{1i}/\eta_1)^{\beta_1}} - 1\right)\left(2e^{-(x_{2i}/\eta_2)^{\beta_2}} - 1\right)} = 0,$$

$$\frac{\beta_2(\sum_{i=1}^n x_{2i}^{\beta_2})}{\eta_2^{\beta_2+1}} - \frac{n\beta_2}{\eta_2} + \sum_{i=1}^n \frac{2\theta\beta_2 \frac{x_{2i}^{\beta_2}}{\eta_2^{\beta_2+1}} e^{-(x_{2i}/\eta_2)^{\beta_2}} \left(2e^{-(x_{1i}/\eta_1)^{\beta_1}} - 1\right)}{1 + \theta\left(2e^{-(x_{1i}/\eta_1)^{\beta_1}} - 1\right)\left(2e^{-(x_{2i}/\eta_2)^{\beta_2}} - 1\right)} = 0. \quad (6)$$

The above system of non-linear equations is solved using Newton's iterative method, with the detailed derivations provided in the Appendix. In practice, obtaining reliable starting values is crucial for convergence. We recommend initializing the algorithm with method-of-moments estimates or estimates from fitting the marginal Weibull distributions independently, which has proven stable in our simulation studies.

3.2 Confidence interval for P_L

In cases where the theoretical distribution of a target statistic is analytically intractable, the asymptotic distribution provides a practical approach for constructing confidence intervals for complex parameters. Accordingly, we pursue interval estimation of P_L based on the asymptotic behavior of its MLE.

Let $\widehat{\gamma} = \widehat{\gamma}_n(X_1, X_2, \ldots, X_n)$ denote the MLE of the parameter γ. Under standard regularity conditions, the estimator $\widehat{\gamma}$ is asymptotically normally distributed as $\sqrt{n}(\widehat{\gamma} - \gamma)$ converges in distribution to a normal distribution with mean zero and variance equal to the Fisher information $I_X(\gamma)$:

$$\sqrt{n}(\widehat{\gamma} - \gamma) \xrightarrow{d} N\big(0, \quad \frac{1}{I_X(\gamma)}\big).$$

The Fisher information $I_X(\gamma)$ obtained as $I_X(\gamma) = E(\frac{\partial}{\partial\gamma} \log f_X(x, \gamma))^2$. Given the parameter vector $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_p)$, the Fisher information matrix $\boldsymbol{I}_X(\boldsymbol{\gamma})$ is expressed as $\boldsymbol{I}_X(\boldsymbol{\gamma}) = [I_{ij}]$, where $I_{ij} = E(\frac{\partial^2}{\partial\gamma_i\partial\gamma_j} \log f_X(x, \boldsymbol{\gamma}))$. This matrix serves as an estimate for the asymptotic variance-covariance structure of the MLE. So,

$$\sqrt{n}(\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}) \xrightarrow{d} N\big(\mathbf{0}, \quad \boldsymbol{I}_X^{-1}(\boldsymbol{\gamma})\big).$$

Consequently, the vector $(\widehat{\beta_1}, \widehat{\beta_2}, \widehat{\eta_1}, \widehat{\eta_2})'$ follows an asymptotic multivariate normal distribution, which is characterized by

$$\sqrt{n}\Big(\begin{pmatrix} \widehat{\beta_1} \\ \widehat{\beta_2} \\ \widehat{\eta_1} \\ \widehat{\eta_2} \end{pmatrix} - \begin{pmatrix} \beta_1 \\ \beta_2 \\ \eta_1 \\ \eta_2 \end{pmatrix} \Big) \xrightarrow{d} N\big(\mathbf{0}, \quad \boldsymbol{I}_X^{-1}(\beta_1, \beta_2, \eta_1, \eta_2)\big),$$

where $\boldsymbol{I}_X^{-1}(\beta_1, \beta_2, \eta_1, \eta_2)$ denotes the inverse of the Fisher information matrix

$$\boldsymbol{I}_X(\beta_1,\beta_2,\eta_1,\eta_2) = \begin{pmatrix} -E(\frac{\partial^2}{\partial\beta_1^2}l) & -E(\frac{\partial^2}{\partial\beta_1\partial\beta_2}l) & -E(\frac{\partial^2}{\partial\beta_1\partial\eta_1}l) & -E(\frac{\partial^2}{\partial\beta_1\partial\eta_2}l) \\ -E(\frac{\partial^2}{\partial\beta_2\partial\beta_1}l) & -E(\frac{\partial^2}{\partial\beta_2^2}l) & -E(\frac{\partial^2}{\partial\beta_2\partial\eta_1}l) & -E(\frac{\partial^2}{\partial\beta_2\partial\eta_2}l) \\ -E(\frac{\partial^2}{\partial\eta_1\partial\beta_1}l) & -E(\frac{\partial^2}{\partial\eta_1\partial\beta_2}l) & -E(\frac{\partial^2}{\partial\eta_1^2}l) & -E(\frac{\partial^2}{\partial\eta_1\partial\eta_2}l) \\ -E(\frac{\partial^2}{\partial\eta_2\partial\beta_1}l) & -E(\frac{\partial^2}{\partial\eta_2\partial\beta_2}l) & -E(\frac{\partial^2}{\partial\eta_2\partial\eta_1}l) & -E(\frac{\partial^2}{\partial\eta_2^2}l) \end{pmatrix},$$

where l represents the log-likelihood function, as specified in (10) of the Appendix.

Given that $P_L = h(\beta_1,\beta_2,\eta_1,\eta_2)$, the invariance property of the MLE implies that $\widehat{P}_L = h(\widehat{\beta_1},\widehat{\beta_2},\widehat{\eta_1},\widehat{\eta_2})$. Since an exact finite-sample distribution for $\widehat{P}_L$ is intractable, we rely on its asymptotic distribution for inference. Consequently, the delta method is employed to approximate the distribution of $\widehat{P}_L$, i.e.

$$\sqrt{n}(\widehat{P}_L - P_L) \xrightarrow{d} N\big(0, Var(\widehat{P}_L)\big),$$

where

$$Var(\widehat{P}_L) = A'\boldsymbol{I}_X^{-1}A. \tag{7}$$

The vector A consists of the partial derivatives of P_L with respect to the parameters β_1, β_2, η_1 and η_2, as specified in (14) in the Appendix. Accordingly,

$$\frac{\sqrt{n}(\widehat{P}_L - P_L)}{\sqrt{Var(\widehat{P}_L)}} \xrightarrow{d} N(0,1).$$

Naturally, estimating the variance of $\widehat{P}_L$ is essential for conducting statistical inference. A consistent estimator for this variance is denoted by $\widehat{Var(\widehat{P}_L)}$. Hence,

$$\frac{\sqrt{n}(\widehat{P}_L - P_L)}{\sqrt{\widehat{Var(\widehat{P}_L)}}} \xrightarrow{d} N(0,1), \tag{8}$$

where

$$\widehat{Var(\widehat{P}_L)} = A'\boldsymbol{I}_o^{-1}(\widehat{\beta_1},\widehat{\beta}_2,\widehat{\eta}_1,\widehat{\eta}_2)A_{\left\{\substack{\beta_1=\widehat{\beta}_1,\ \beta_2=\widehat{\beta}_2 \\ \eta_1=\widehat{\eta}_1,\ \eta_2=\widehat{\eta}_2}\right.}.$$

The observed Fisher information matrix is as given bellow

$$\boldsymbol{I}_o(\widehat{\beta}_1,\widehat{\beta}_2,\widehat{\eta}_1,\widehat{\eta}_2) = \begin{bmatrix} -\frac{\partial^2}{\partial\beta_1^2}l & -\frac{\partial^2}{\partial\beta_1\partial\beta_2}l & -\frac{\partial^2}{\partial\beta_1\partial\eta_1}l & -\frac{\partial^2}{\partial\beta_1\partial\eta_2}l \\ -\frac{\partial^2}{\partial\beta_2\partial\beta_1}l & -\frac{\partial^2}{\partial\beta_2^2}l & -\frac{\partial^2}{\partial\beta_2\partial\eta_1}l & -\frac{\partial^2}{\partial\beta_2\partial\eta_2}l \\ -\frac{\partial^2}{\partial\eta_1\partial\beta_1}l & -\frac{\partial^2}{\partial\eta_1\partial\beta_2}l & -\frac{\partial^2}{\partial\eta_1^2}l & -\frac{\partial^2}{\partial\eta_1\partial\eta_2}l \\ -\frac{\partial^2}{\partial\eta_2\partial\beta_1}l & -\frac{\partial^2}{\partial\eta_2\partial\beta_2}l & -\frac{\partial^2}{\partial\eta_2\partial\eta_1}l & -\frac{\partial^2}{\partial\eta_2^2}l \end{bmatrix}_{\left\{\substack{\beta_1=\widehat{\beta}_1,\ \beta_2=\widehat{\beta}_2 \\ \eta_1=\widehat{\eta}_1,\ \eta_2=\widehat{\eta}_2}\right.}. \tag{9}$$

The components of the observed information matrix $\boldsymbol{I}_o$ are detailed in (19) in the Appendix.

The upper confidence limit for the probability of non-conforming products serves as a valuable metric. Manufacturers often use this bound as a benchmark for assuring product quality to customers (Shiau et al., 2012). Based on (8), the one-sided $100(1-\alpha)\%$ confidence interval for P_L can be derived using the following formula:

$$p\Big(\frac{\sqrt{n}(\widehat{P}_L - P_L)}{\sqrt{\widehat{Var(\widehat{P}_L)}}} \leq z_\alpha\Big) = 1-\alpha.$$

Accordingly, the $100(1-\alpha)\%$ confidence interval for P_L is given by the following expression.

$$\Big(0, \quad \widehat{P}_L + z_\alpha \sqrt{\frac{\widehat{Var(\widehat{P}_L)}}{n}}\Big).$$

3.3 Hypothesis testing for the non-conforming rate

One of the key issues in statistical inference involves testing hypotheses regarding population parameters. In this context, we aim to test the null hypothesis $H_0: \ P_L \geq p_0$, which indicates that the process is not capable, against the alternative hypothesis $H_1: \ P_L < p_0$, signifying that the process meets the capability requirements.

Due to the analytical complexity, it is generally infeasible to express P_L explicitly in terms of C_{L_1} and C_{L_2}. Consequently, for a given threshold p_0, (4) yields multiple feasible combinations of β_1, β_2, η_1 and η_2, where the choice among them depends on the practical significance of the problem. Given the intractable distribution of $\widehat{P}_L$, a Monte Carlo simulation approach is utilized to support decision-making under uncertainty.

The following simulation-based procedure is proposed to perform hypothesis testing on the capability of the process:
Initially, a random sample of size n is collected from the process outputs, and the Kendall rank correlation coefficient is calculated. Using the relationship $\widehat{\theta} = 9\tau/2$, an estimate of the copula parameter is derived. Next, the MLEs of the marginal parameters β_1, β_2, η_1 and η_2 are obtained by solving the non-linear system in (6), and then substituted into (5) to estimate the non-conforming probability $\widehat{P}_L$, denoted by $\widehat{P}_{L_0}$. The Monte Carlo simulation steps are:

- Step 1: Determine the significance level α for the hypothesis test.
- Step 2: Identify a set of parameters β_1, β_2, η_1, η_2 such that $P_L = p_0$.
- Step 3: Simulate a sample of size n from the bivariate Weibull distribution with FGM copula using β_1, β_2, η_1, η_2 and $\widehat{\theta}$.
- Step 4: Estimate the parameters $\widehat{\beta}_1$, $\widehat{\beta}_2$, $\widehat{\eta}_1$, $\widehat{\eta}_2$ and compute $\widehat{P}_L$.
- Step 5: Repeat steps 3 and 4 a total of m times.
- Step 6: Count the number of simulations, r, for which the estimated test statistic $\widehat{P}_L$ is less than $\widehat{P}_{L_0}$.
- Step 7: The estimated p-value is computed as $\hat{p} = r/m$. If $\hat{p} < \alpha$, the null hypothesis is rejected, indicating that the process meets the required capability level. Otherwise, there is insufficient evidence to reject the null hypothesis.

4 Numerical illustration

Consider a production process in which the lifetimes of products are governed by two dependent characteristics modeled via a bivariate Weibull distribution with the FGM copula framework. A product is deemed conforming if the lifetime of the first characteristic exceeds 3 units, and that of the second characteristic exceeds 2.5 units.

A random sample of size 50 was collected, with the data summarized in Table 1. Separate goodness-of-fit tests were conducted on each marginal distribution of the bivariate sample, leading to the conclusion that both marginal distributions follow the Weibull distribution.

Table 1: Collected lifetime data for the manufactured products.

case	sample data	case	sample data	case	sample data
1	(8.1846, 12.4304)	18	(9.7517, 9.2832)	35	(6.8669, 5.3385)
2	(9.0195, 10.2305)	19	(8.8896, 12.9511)	36	(8.7949, 9.6006)
3	(8.4517, 11.1600)	20	(7.2251, 12.8749)	37	(8.3157, 12.0161)
4	(6.3147, 9.5100)	21	(8.9808, 7.1559)	38	(9.0985, 11.3870)
5	(7.3522, 6.5469)	22	(8.5608, 4.4536)	39	(8.7445, 10.1003)
6	(7.6041, 9.0294)	23	(7.7739, 9.8530)	40	(8.6225, 8.9122)
7	(8.2137, 4.7651)	24	(5.3440, 3.3664)	41	(7.4324, 6.7555)
8	(10.1698, 7.4616)	25	(8.1119, 7.4021)	42	(9.1734, 12.5307)
9	(8.1214, 10.4171)	26	(8.7686, 6.6247)	43	(8.6874, 12.6640)
10	(6.3490, 7.0637)	27	(8.3248, 9.2624)	44	(7.8702, 1.1515)
11	(8.5380, 14.6689)	28	(8.5280, 6.2132)	45	(5.7563, 10.0242)
12	(9.1202, 6.9574)	29	(8.9842, 9.3705)	46	(7.4756, 9.5415)
13	(7.1395, 8.8916)	30	(8.8993, 11.1663)	47	(9.2709, 8.9578)
14	(8.2562, 4.8949)	31	(7.5585, 6.4648)	48	(9.0689, 9.2311)
15	(8.0172, 8.8154)	32	(6.2924, 4.4632)	49	(4.6879, 9.2889)
16	(8.2258, 9.0366)	33	(9.2389, 9.3259)	50	(7.5845, 8.5297)
17	(8.0155, 8.9844)	34	(8.4667, 3.4729)		

In order to assess whether the dependence structure follows the FGM copula model, the dependence parameter along with other model parameters were first estimated. To quantify the degree of association between paired observations, the sample value of Kendall's tau was calculated, yielding $\tau = 0.177$. Consequently, the corresponding dependence parameter was estimated as $\widehat{\theta} = 0.797$. Utilizing the system of non-linear (6), the MLEs of the parameters were obtained as $\widehat{\beta}_1 = 9.54$, $\widehat{\beta}_2 = 3.47$, $\widehat{\eta}_1 = 8.52$ and $\widehat{\eta}_2 = 9.51$. Accordingly, the estimated non-conforming probability was computed as $\widehat{P}_L = 0.0096$.

Subsequently, 500 observations were simulated from a bivariate Weibull distribution with dependence model by the FGM copula model. Figure 1 displays scatter plots for both the original paired sample data and the simulated data, along with the empirical and theoretical distribution representations. The empirical distribution is illustrated by the points $(R_i/(n+1), S_i/(n+1))$, where R_i and S_i denote the ranks corresponding to the observed sample. In contrast, the theoretical distribution is derived from the marginal distributions of the simulated 500 paired observations generated under the assumed FGM copula-based bivariate Weibull model. In both plots, the original sample points are marked by red stars for distinction.

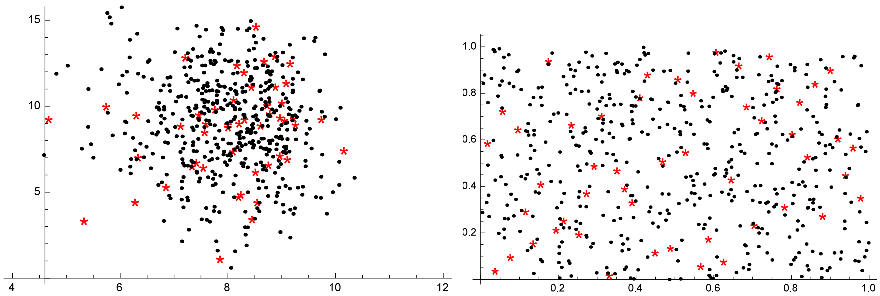

Figure 1: Scatter plots of 500 simulated observations from the bivariate Weibull model with the FGM copula (left panel), and the corresponding theoretical distribution (right panel).

Figure 2 presents both the chi-plot and the Kendall plot (K-plot) corresponding to the data. Visually, these diagnostic tools suggest the presence of a positive dependence structure between the two variables. Moreover, the plots indicate an absence of tail dependence, implying that the extremal behavior of the variables is not strongly correlated.

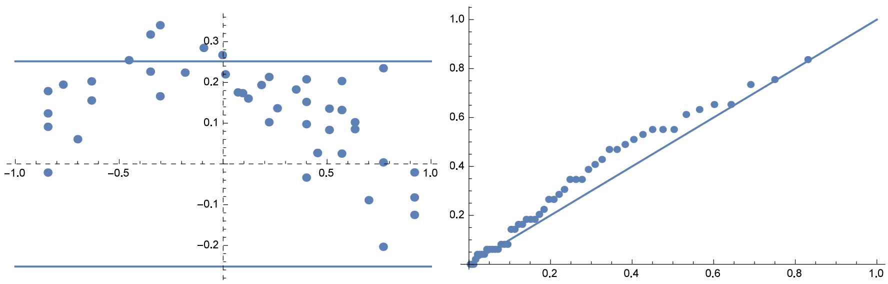

Figure 2: Goodness-of-fit assessment via chi-plot (left panel) and K-plot (right panel).

The collective evidence from the aforementioned dependence measures and graphical analyses supports the conclusion that the dependence structure of the data is well characterized by the FGM copula model. Additionally, the goodness-of-fit test based on the Cramér-von Mises statistic yielded a value of 0.0104, with a corresponding p-value of 0.9995. This strongly confirms that the FGM copula provides an appropriate and accurate representation of the underlying dependence structure.

Moreover, the 0.95% one-sided confidence interval for the non-conforming rate is obtained as $(0,\ 0.0096 + 0.006 z_{0.05}) = (0, 0.019)$.

Additionally, we employ a Monte Carlo simulation procedure to examine the hypothesis testing problems $H_0 :\ P_L \geq 0.01$ versus $H_1 :\ P_L < 0.01$, and $H_0 :\ P_L \geq 0.03$ versus $H_1 :\ P_L < 0.03$. Table 2 presents selected parameter values for β_1, β_2, η_1 and η_2 that satisfy the corresponding null hypotheses. For each scenario, 1000 samples of size 50 were generated, and the associated simulated p-values were computed. The results indicate that for all examined cases under $H_0 :\ P_L \geq 0.01$, the simulated p-values exceed the significance level $\alpha = 0.05$, leading to the conclusion that the null hypothesis

Table 2: Weibull parameter sets under H_0 and the resulting simulated p-values.

case	$H_0:\ P_L \geq 0.01$		$H_0:\ P_L \geq 0.03$	
	$(\beta_1, \beta_2, \eta_1, \eta_2)$	Simulated PV	$(\beta_1, \beta_2, \eta_1, \eta_2)$	Simulated PV
1	(6.50, 16.39, 9.91, 3.31)	0.482	(9.61, 4.22, 13.60, 5.72)	0.025
2	(5.77, 4.73, 9.16, 6.80)	0.450	(9.90, 7.04, 6.68, 4.11)	0.018
3	(3.77, 1.93, 11.85, 41.07)	0.461	(6.02, 5.42, 14.69, 4.76)	0.021
4	(9.28, 7.64, 9.36, 4.56)	0.508	(8.55, 2.44, 9.19, 10.44)	0.024
5	(9.67, 6.37, 10.94, 5.14)	0.521	(9.28, 12.15, 5.87, 3.35)	0.020
6	(1.80, 3.90, 40.00, 12.00)	0.433	(9.52, 3.43, 8.25, 6.92)	0.025
7	(4.58, 5.15, 8.86, 7.63)	0.478	(3.99, 14.60, 7.27, 3.93)	0.015
8	(4.18, 1.41, 12.06, 83.31)	0.525	(10.20, 5.30, 11.12, 4.83)	0.015
9	(5.93, 5.96, 6.89, 6.36)	0.391	(5.72, 7.38, 11.51, 4.02)	0.014
10	(7.65, 4.74, 14.94, 6.59)	0.535	(7.16, 8.21, 14.26, 3.82)	0.032
11	(5.92, 19.36, 12.70, 3.17)	0.511	(2.85, 10.91, 12.09, 3.75)	0.002
12	(9.92, 11.96, 10.97, 3.67)	0.513	(4.05, 6.33, 8.02, 5.02)	0.005
13	(7.65, 11.96, 12.25, 3.65)	0.469	(5.24, 5.89, 7.71, 4.72)	0.011
14	(6.33, 19.92, 14.65, 3.15)	0.526	(7.53, 7.41, 7.91, 4.01)	0.016
15	(9.62, 3.24, 12.26, 10.34)	0.545	(2.92, 5.45, 12.94, 5.30)	0.018
16	(9.91, 10.49, 10.06, 3.87)	0.530	(9.29, 17.36, 5.76, 3.07)	0.016
17	(6.65, 13.36, 11.63, 3.53)	0.517	(7.96, 11.21, 11.76, 3.41)	0.018
18	(6.79, 12.47, 7.61, 3.67)	0.490	(5.64, 5.59, 7.14, 4.90)	0.015
19	(7.96, 6.27, 5.37, 8.91)	0.534	(8.02, 7.43, 12.80, 3.99)	0.018
20	(9.08, 9.22, 8.50, 4.12)	0.515	(8.47, 10.62, 8.94, 3.47)	0.012
21	(5.38, 3.18, 8.85, 11.80)	0.494	(8.97, 19.50, 14.31, 2.99)	0.026
22	(8.27, 5.68, 12.01, 5.62)	0.532	(3.11, 6.46, 10.51, 5.07)	0.007
23	(3.29, 7.19, 12.33, 7.20)	0.495	(6.61, 1.62, 9.22, 21.78)	0.030
24	(5.86, 2.60, 13.51, 14.74)	0.514	(6.76, 2.07, 10.21, 13.55)	0.019
25	(7.67, 17.51, 9.26, 3.25)	0.506	(2.63, 3.03, 13.25, 11.22)	0.003
26	(8.43, 12.90, 14.67, 3.57)	0.516	(5.39, 6.49, 9.41, 4.32)	0.013
27	(7.65, 8.49, 9.47, 4.30)	0.520	(2.93, 14.85, 13.37, 3.27)	0.005
28	(4.75, 13.95, 14.10, 3.49)	0.526	(5.06, 6.65, 11.86, 4.24)	0.024
29	(5.18, 6.19, 10.30, 5.41)	0.526	(7.02, 1.82, 13.49, 17.02)	0.024
30	(4.56, 3.39, 13.68, 9.95)	0.498	(6.58, 7.12, 14.93, 4.08)	0.023

cannot be rejected. In contrast, for all tested cases under $H_0:\ P_L \geq 0.03$, the p-values fall below $\alpha = 0.05$, and thus, the null hypothesis is rejected in favor of the alternative.

5 Conclusion

This study focused on modeling the non-conforming rate (P_L) in production processes where product lifetimes depend on two interrelated characteristics governed by a bivariate Weibull distribution structured through the FGM copula. The MLE for P_L was derived and studied. Due to the analytical intractability of the exact sampling distribution of $\widehat{P}_L$, its asymptotic distribution was applied to construct confidence intervals. In addition, a Monte Carlo simulation approach was adopted to facilitate hypothesis testing regarding P_L. To illustrate the practical implementation, a comprehensive numerical example was included.

This study has developed an inference framework for the non-conforming rate in bivariate lifetime data using the FGM copula with Weibull margins. It is important to acknowledge the primary limitation of this approach: the FGM copula is restricted to modeling weak-to-moderate dependence and lacks tail dependence. Consequently, the proposed model is best suited for systems where component failures are not strongly correlated or driven by common extreme shocks. For applications involving stronger dependence or tail-dependent patterns, future research could adapt the presented methodology to more flexible copula families, such as the Clayton, Gumbel, or elliptical copulas. Empirical copula selection tests and goodness-of-fit procedures would be natural extensions in such applied settings.

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Appendix

To obtain (5), one must first compute the MLEs of the four parameters β_1, β_2, η_1 and η_2. The joint pdf of the bivariate Weibull distribution under the FGM copula structure is given by

$$\begin{aligned} f_{X_1,X_2}(x_1,x_2,\beta_1,\beta_2,\eta_1,\eta_2,\theta) &= \frac{\beta_1}{\eta_1}(\frac{x_1}{\eta_1})^{\beta_1-1}e^{-(\frac{x_1}{\eta_1})^{\beta_1}}\frac{\beta_2}{\eta_2}(\frac{x_2}{\eta_2})^{\beta_2-1}e^{-(\frac{x_2}{\eta_2})^{\beta_2}} \\ &\quad \times\Big[1+\theta\big(2e^{-(\frac{x_1}{\eta_1})^{\beta_1}}-1\big)\big(2e^{-(\frac{x_2}{\eta_2})^{\beta_2}}-1\big)\Big]. \end{aligned}$$

Accordingly, the likelihood function corresponding to a single observation (i.e., for a sample size of $n = 1$) takes the form

$$\begin{aligned} L(\beta_1,\beta_2,\eta_1,\eta_2,x_1,x_2) &= \frac{\beta_1}{\eta_1}(\frac{x_1}{\eta_1})^{\beta_1-1}e^{-(\frac{x_1}{\eta_1})^{\beta_1}}\frac{\beta_2}{\eta_2}(\frac{x_2}{\eta_2})^{\beta_2-1}e^{-(\frac{x_2}{\eta_2})^{\beta_2}} \\ &\quad \times\Big[1+\theta\big(2e^{-(\frac{x_1}{\eta_1})^{\beta_1}}-1\big)\big(2e^{-(\frac{x_2}{\eta_2})^{\beta_2}}-1\big)\Big], \end{aligned}$$

Consequently, the corresponding log-likelihood function is obtained as follows

$$\begin{aligned} l(\beta_1,\beta_2,\eta_1,\eta_2,x_1,x_2) &= \log\beta_1-\log\eta_1+(\beta_1-1)\log(\frac{x_1}{\eta_1})-(\frac{x_1}{\eta_1})^{\beta_1}+\log\beta_2 \\ &\quad -\log\eta_2+(\beta_2-1)\log(\frac{x_2}{\eta_2})-(\frac{x_2}{\eta_2})^{\beta_2} \\ &\quad +\log\Big[1+\theta\big(2e^{-(\frac{x_1}{\eta_1})^{\beta_1}}-1\big)\big(2e^{-(\frac{x_2}{\eta_2})^{\beta_2}}-1\big)\Big]. \end{aligned} \tag{10}$$

By differentiating (10) with respect to each of the parameters β_1, β_2, η_1 and η_2, and equating the resulting expressions to zero, the following system of non-linear equations is derived:

$$\frac{1}{\beta_1}+\log(\frac{x_1}{\eta_1})-(\frac{x_1}{\eta_1})^{\beta_1}\log(\frac{x_1}{\eta_1})-\frac{2\theta(\frac{x_1}{\eta_1})^{\beta_1}\log(\frac{x_1}{\eta_1})e^{-(x_1/\eta_1)^{\beta_1}}\big(2e^{-(x_2/\eta_2)^{\beta_2}}-1\big)}{1+\theta\big(2e^{-(x_1/\eta_1)^{\beta_1}}-1\big)\big(2e^{-(x_2/\eta_2)^{\beta_2}}-1\big)}=0,$$

$$\frac{1}{\beta_2}+\log(\frac{x_2}{\eta_2})-(\frac{x_2}{\eta_2})^{\beta_2}\log(\frac{x_2}{\eta_2})-\frac{2\theta(\frac{x_2}{\eta_2})^{\beta_2}\log(\frac{x_2}{\eta_2})e^{-(x_2/\eta_2)^{\beta_2}}\big(2e^{-(x_1/\eta_1)^{\beta_1}}-1\big)}{1+\theta\big(2e^{-(x_1/\eta_1)^{\beta_1}}-1\big)\big(2e^{-(x_2/\eta_2)^{\beta_2}}-1\big)}=0,$$

$$\frac{\beta_1}{\eta_1}(\frac{x_1}{\eta_1})^{\beta_1}-\frac{\beta_1}{\eta_1}+\frac{2\theta\frac{\beta_1}{\eta_1}(\frac{x_1}{\eta_1})^{\beta_1}e^{-(x_1/\eta_1)^{\beta_1}}\big(2e^{-(x_2/\eta_2)^{\beta_2}}-1\big)}{1+\theta\big(2e^{-(x_1/\eta_1)^{\beta_1}}-1\big)\big(2e^{-(x_2/\eta_2)^{\beta_2}}-1\big)}=0,$$

$$\frac{\beta_2}{\eta_2}(\frac{x_2}{\eta_2})^{\beta_2}-\frac{\beta_2}{\eta_2}+\frac{2\theta\frac{\beta_2}{\eta_2}(\frac{x_2}{\eta_2})^{\beta_2}e^{-(x_2/\eta_2)^{\beta_2}}\big(2e^{-(x_1/\eta_1)^{\beta_1}}-1\big)}{1+\theta\big(2e^{-(x_1/\eta_1)^{\beta_1}}-1\big)\big(2e^{-(x_2/\eta_2)^{\beta_2}}-1\big)}=0. \qquad (11)$$

Accordingly, the MLEs of the parameters are computed by solving the system of non-linear equations given in (11) using Newton's iterative method.

The joint pdf based on a random sample of size n, denoted by $\underline{x}=(x_1,x_2,\ldots,x_n)'$, where each observation is of the form $x_i=(x_{1i},x_{2i})$, is expressed as

$$\begin{aligned} f_{\underline{X}}(\underline{x},\beta_1,\beta_2,\eta_1,\eta_2,\theta) &= (\frac{\beta_1}{\eta_1})^n\frac{\big(\prod_{i=1}^n x_{1i}\big)^{\beta_1-1}}{\eta_1^{n(\beta_1-1)}}e^{-(\sum_{i=1}^n x_{1i}^{\beta_1})/\eta_1^{\beta_1}}(\frac{\beta_2}{\eta_2})^n \\ &\times\frac{\big(\prod_{i=1}^n x_{2i}\big)^{\beta_2-1}}{\eta_2^{n(\beta_2-1)}}e^{-(\sum_{i=1}^n x_{2i}^{\beta_2})/\eta_2^{\beta_2}} \\ &\prod_{i=1}^n\Big[1+\theta\big(2e^{-(x_{1i}/\eta_1)^{\beta_1}}-1\big)\big(2e^{-(x_{2i}/\eta_2)^{\beta_2}}-1\big)\Big]. \end{aligned} \qquad (12)$$

First, the logarithm of (12) is taken. Then, partial derivatives with respect to the parameters β_1, β_2, η_1 and η_2 are computed and set to zero. Using Newton's iterative method, the estimates $\widehat{\beta}_1$, $\widehat{\beta}_2$, $\widehat{\eta}_1$ and $\widehat{\eta}_2$ are obtained by solving the resulting system of nonlinear (6). Finally, (5) is evaluated.

We now derive the components of vector A given in (7). Noting that $P_L = h(\beta_1,\beta_2,\eta_1,\eta_2)$, we differentiate P_L, as defined in (4), with respect to the parameters β_1, β_2, η_1 and η_2.

Define the vector

$$A=(\frac{\partial h}{\partial\beta_1},\frac{\partial h}{\partial\beta_2},\frac{\partial h}{\partial\eta_1},\frac{\partial h}{\partial\eta_2})'. \qquad (13)$$

Accordingly, we obtain

$$\begin{aligned} \frac{\partial h}{\partial\beta_1} &= (\frac{L_1}{\eta_1})^{\beta_1}\log(\frac{L_1}{\eta_1})e^{-(L_1/\eta_1)^{\beta_1}} && (14) \\ &\Big[1-\big(1-e^{-(l_2/\eta_2)^{\beta_2}}\big)\big(1+2\theta e^{-(l_1/\eta_1)^{\beta_1}-(l_2/\eta_2)^{\beta_2}}-\theta e^{-(l_2/\eta_2)^{\beta_2}}\big)\Big], \\ \frac{\partial h}{\partial\beta_2} &= (\frac{L_2}{\eta_2})^{\beta_2}\log(\frac{L_2}{\eta_2})e^{-(L_2/\eta_2)^{\beta_2}} && (15) \\ &\Big[1-\big(1-e^{-(l_1/\eta_1)^{\beta_1}}\big)\big(1+2\theta e^{-(l_1/\eta_1)^{\beta_1}-(l_2/\eta_2)^{\beta_2}}-\theta e^{-(l_1/\eta_1)^{\beta_1}}\big)\Big], \\ \frac{\partial h}{\partial\eta_1} &= \frac{\beta_1}{\eta_1}(\frac{L_1}{\eta_1})^{\beta_1}e^{-(L_1/\eta_1)^{\beta_1}} && (16) \end{aligned}$$

$$\Big[\big(1-e^{-(l_2/\eta_2)^{\beta_2}}\big)\big(1+2\theta e^{-(l_1/\eta_1)^{\beta_1}-(l_2/\eta_2)^{\beta_2}}-\theta e^{-(l_2/\eta_2)^{\beta_2}}\big)-1\Big],$$

$$\frac{\partial h}{\partial \eta_2} = \frac{\beta_2}{\eta_2}(\frac{L_2}{\eta_2})^{\beta_2}e^{-(L_2/\eta_2)^{\beta_2}} \tag{17}$$

$$\Big[\big(1-e^{-(l_1/\eta_1)^{\beta_1}}\big)\big(1+2\theta e^{-(l_1/\eta_1)^{\beta_1}-(l_2/\eta_2)^{\beta_2}}-\theta e^{-(l_1/\eta_1)^{\beta_1}}\big)-1\Big]. \tag{18}$$

To compute the elements of the observed information matrix in (9), we define

$$\boldsymbol{I}_o(\widehat{\beta}_1,\widehat{\beta}_2,\widehat{\eta}_1,\widehat{\eta}_2)=\begin{pmatrix} M_{11} & M_{12} & M_{13} & M_{14}\\ M_{21} & M_{22} & M_{23} & M_{24}\\ M_{31} & M_{32} & M_{33} & M_{34}\\ M_{41} & M_{42} & M_{43} & M_{44}\end{pmatrix}, \tag{19}$$

in which

$$M_{11}=\frac{1}{\widehat{\beta}_1^2}+(\frac{x_1}{\widehat{\eta}_1})^{\widehat{\beta}_1}\log^2(\frac{x_1}{\widehat{\eta}_1})+\frac{A_{11}}{\big[1+\theta\big(2e^{-(x_1/\widehat{\eta}_1)^{\widehat{\beta}_1}}-1\big)\big(2e^{-(x_2/\widehat{\eta}_2)^{\widehat{\beta}_2}}-1\big)\big]^2},$$

$$A_{11}=2\theta(\frac{x_1}{\widehat{\eta}_1})^{\widehat{\beta}_1}\log^2(\frac{x_1}{\widehat{\eta}_1})e^{-(x_1/\widehat{\eta}_1)^{\widehat{\beta}_1}}\big(2e^{-(x_2/\widehat{\eta}_2)^{\widehat{\beta}_2}}-1\big)\Big\{\big(1-(\frac{x_1}{\widehat{\eta}_1})^{\widehat{\beta}_1}\big)$$

$$\times\Big[1+\theta\big(2e^{-(x_1/\widehat{\eta}_1)^{\widehat{\beta}_1}}-1\big)\big(2e^{-(x_2/\widehat{\eta}_2)^{\widehat{\beta}_2}}-1\big)\Big]$$

$$+2\theta(\frac{x_1}{\widehat{\eta}_1})^{\widehat{\beta}_1}e^{-(x_1/\widehat{\eta}_1)^{\widehat{\beta}_1}}\big(2e^{-(x_2/\widehat{\eta}_2)^{\widehat{\beta}_2}}-1\big)\Big\},$$

$$M_{12}=\frac{A_{12}}{\big[1+\theta\big(2e^{-(x_1/\widehat{\eta}_1)^{\widehat{\beta}_1}}-1\big)\big(2e^{-(x_2/\widehat{\eta}_2)^{\widehat{\beta}_2}}-1\big)\big]^2},$$

$$A_{12}=4\theta(\frac{x_1}{\widehat{\eta}_1})^{\widehat{\beta}_1}(\frac{x_2}{\widehat{\eta}_2})^{\widehat{\beta}_2}\log(\frac{x_1}{\widehat{\eta}_1})\log(\frac{x_2}{\widehat{\eta}_2})e^{-(x_1/\widehat{\eta}_1)^{\widehat{\beta}_1}-(x_2/\widehat{\eta}_2)^{\widehat{\beta}_2}}$$

$$\times\Big[(1-\theta)\big(2e^{-(x_1/\widehat{\eta}_1)^{\widehat{\beta}_1}}-1\big)\big(2e^{-(x_2/\widehat{\eta}_2)^{\widehat{\beta}_2}}-1\big)-1\Big],$$

$$M_{13}=\frac{1}{\widehat{\eta}_1}-\frac{1}{\widehat{\eta}_1}(\frac{x_1}{\widehat{\eta}_1})^{\widehat{\beta}_1}[1+\widehat{\beta}_1\log(\frac{x_1}{\widehat{\eta}_1})]+\frac{A_{13}}{\big[1+\theta\big(2e^{-(x_1/\widehat{\eta}_1)^{\widehat{\beta}_1}}-1\big)\big(2e^{-(x_2/\widehat{\eta}_2)^{\widehat{\beta}_2}}-1\big)\big]^2},$$

$$A_{13}=2\theta x_1^{\widehat{\beta}_1}e^{-(x_1/\widehat{\eta}_1)^{\widehat{\beta}_1}}\big(2e^{-(x_2/\widehat{\eta}_2)^{\widehat{\beta}_2}}-1\big)\big[1+\theta\big(2e^{-(x_1/\widehat{\eta}_1)^{\widehat{\beta}_1}}-1\big)\big(2e^{-(x_2/\widehat{\eta}_2)^{\widehat{\beta}_2}}-1\big)\big]$$

$$\times\Big[-\frac{1}{\widehat{\eta}_1^{\widehat{\beta}_1+1}}\big[1+\widehat{\beta}_1\log(\frac{x_1}{\widehat{\eta}_1})\big]+\widehat{\beta}_1\frac{x_1^{\widehat{\beta}_1}}{\widehat{\eta}_1^{2\widehat{\beta}_1+1}}\log(\frac{x_1}{\widehat{\eta}_1})\Big]-4\theta^2\frac{\widehat{\beta}_1}{\widehat{\eta}_1}(\frac{x_1}{\widehat{\eta}_1})^{2\widehat{\beta}_1}e^{-(x_1/\widehat{\eta}_1)^{\widehat{\beta}_1}}$$

$$\times\big(2e^{-(x_2/\widehat{\eta}_2)^{\widehat{\beta}_2}}-1\big)^2\Big[\log(\frac{x_1}{\widehat{\eta}_1})e^{-(x_1/\widehat{\eta}_1)^{\widehat{\beta}_1}}\Big],$$

$$M_{14}=\frac{4\theta\frac{\widehat{\beta}_2}{\widehat{\eta}_2}(\frac{x_1}{\widehat{\eta}_1})^{\widehat{\beta}_1}(\frac{x_2}{\widehat{\eta}_2})^{\widehat{\beta}_2}\log(\frac{x_1}{\widehat{\eta}_1})e^{-(x_1/\widehat{\eta}_1)^{\widehat{\beta}_1}-(x_2/\widehat{\eta}_2)^{\widehat{\beta}_2}}}{\big[1+\theta\big(2e^{-(x_1/\widehat{\eta}_1)^{\widehat{\beta}_1}}-1\big)\big(2e^{-(x_2/\widehat{\eta}_2)^{\widehat{\beta}_2}}-1\big)\big]^2},$$

$$M_{22}=\frac{1}{\widehat{\beta}_2^2}+(\frac{x_2}{\widehat{\eta}_2})^{\widehat{\beta}_2}\log^2(\frac{x_2}{\widehat{\eta}_2})+\frac{A_{22}}{\big[1+\theta\big(2e^{-(x_1/\widehat{\eta}_1)^{\widehat{\beta}_1}}-1\big)\big(2e^{-(x_2/\widehat{\eta}_2)^{\widehat{\beta}_2}}-1\big)\big]^2},$$

$$A_{22}=2\theta(\frac{x_2}{\widehat{\eta}_2})^{\widehat{\beta}_2}\log^2(\frac{x_2}{\widehat{\eta}_2})e^{-(x_2/\widehat{\eta}_2)^{\widehat{\beta}_2}}\big(2e^{-(x_1/\widehat{\eta}_1)^{\widehat{\beta}_1}}-1\big)\Big\{\big(1-(\frac{x_2}{\widehat{\eta}_2})^{\widehat{\beta}_2}\big)$$

$$\begin{aligned}
&\Big[1+\theta\big(2e^{-(x_1/\widehat{\eta}_1)^{\widehat{\beta}_1}}-1\big)\big(2e^{-(x_2/\widehat{\eta}_2)^{\widehat{\beta}_2}}-1\big)\Big]\\
&+2\theta(\frac{x_2}{\widehat{\eta}_2})^{\widehat{\beta}_2}e^{-(x_2/\widehat{\eta}_2)^{\widehat{\beta}_2}}\left(2e^{-(x_1/\widehat{\eta}_1)^{\widehat{\beta}_1}}-1\right)\Big\},\\
M_{23}=&\frac{4\theta\frac{\widehat{\beta}_1}{\widehat{\eta}_1}(\frac{x_1}{\widehat{\eta}_1})^{\widehat{\beta}_1}(\frac{x_2}{\widehat{\eta}_2})^{\widehat{\beta}_2}\log(\frac{x_2}{\widehat{\eta}_2})e^{-(x_1/\widehat{\eta}_1)^{\widehat{\beta}_1}-(x_2/\widehat{\eta}_2)^{\widehat{\beta}_2}}}{\big[1+\theta\big(2e^{-(x_1/\widehat{\eta}_1)^{\widehat{\beta}_1}}-1\big)\big(2e^{-(x_2/\widehat{\eta}_2)^{\widehat{\beta}_2}}-1\big)\big]^2},\\
M_{24}=&\frac{1}{\widehat{\eta}_2}-\frac{1}{\widehat{\eta}_2}(\frac{x_2}{\widehat{\eta}_2})^{\widehat{\beta}_2}[1+\widehat{\beta}_2\log(\frac{x_2}{\widehat{\eta}_2})]+\frac{A_{24}}{\big[1+\theta\big(2e^{-(x_1/\widehat{\eta}_1)^{\widehat{\beta}_1}}-1\big)\big(2e^{-(x_2/\widehat{\eta}_2)^{\widehat{\beta}_2}}-1\big)\big]^2},\\
A_{24}=&2\theta x_2^{\widehat{\beta}_2}e^{-(x_2/\widehat{\eta}_2)^{\widehat{\beta}_2}}\left(2e^{-(x_1/\widehat{\eta}_1)^{\widehat{\beta}_1}}-1\right)\big[1+\theta\big(2e^{-(x_1/\widehat{\eta}_1)^{\widehat{\beta}_1}}-1\big)\\
&\times\big(2e^{-(x_2/\widehat{\eta}_2)^{\widehat{\beta}_2}}-1\big)\big]\Big[-\frac{1}{\widehat{\eta}_2^{\widehat{\beta}_2+1}}\big[1+\widehat{\beta}_2\log(\frac{x_2}{\widehat{\eta}_2})\big]+\widehat{\beta}_2\frac{x_2^{\widehat{\beta}_2}}{\widehat{\eta}_2^{2\widehat{\beta}_2+1}}\log(\frac{x_2}{\widehat{\eta}_2})\Big]\\
&-4\theta^2\frac{\widehat{\beta}_2}{\widehat{\eta}_2}(\frac{x_2}{\widehat{\eta}_2})^{2\widehat{\beta}_2}e^{-(x_2/\widehat{\eta}_2)^{\widehat{\beta}_2}}\left(2e^{-(x_1/\widehat{\eta}_1)^{\widehat{\beta}_1}}-1\right)^2\Big[\log(\frac{x_2}{\widehat{\eta}_2})e^{-(x_2/\widehat{\eta}_2)^{\widehat{\beta}_2}}\Big],\\
M_{33}=&-\frac{n\widehat{\beta}_1}{\widehat{\eta}_1^2}+\frac{\widehat{\beta}_1(\widehat{\beta}_1+1)}{\widehat{\eta}_1^{\widehat{\beta}_1+2}}x_1^{\widehat{\beta}_1}-\frac{A_{33}}{\big[1+\theta\big(2e^{-(x_1/\widehat{\eta}_1)^{\widehat{\beta}_1}}-1\big)\big(2e^{-(x_2/\widehat{\eta}_2)^{\widehat{\beta}_2}}-1\big)\big]^2},\\
A_{33}=&2\theta\widehat{\beta}_1\frac{x_1^{\widehat{\beta}_1}}{\widehat{\eta}_1^{\widehat{\beta}_1+1}}e^{-(x_1/\widehat{\eta}_1)^{\widehat{\beta}_1}}\left(2e^{-(x_2/\widehat{\eta}_2)^{\widehat{\beta}_2}}-1\right)\Big[\frac{1}{\widehat{\eta}_1}\big(\widehat{\beta}_1(\frac{x_1}{\widehat{\eta}_1})^{\widehat{\beta}_1}-(\widehat{\beta}_1+1)\big)\\
&\times\big[1+\theta\big(2e^{-(x_1/\widehat{\eta}_1)^{\widehat{\beta}_1}}-1\big)\times\big(2e^{-(x_2/\widehat{\eta}_2)^{\widehat{\beta}_2}}-1\big)\big]-\frac{x_1^{\widehat{\beta}_1}}{\widehat{\eta}_1^{\widehat{\beta}_1+1}}e^{-(x_1/\widehat{\eta}_1)^{\widehat{\beta}_1}}\Big],\\
M_{34}=&-\frac{4\theta\widehat{\beta}_1\widehat{\beta}_2\frac{x_1^{\widehat{\beta}_1}x_2^{\widehat{\beta}_2}}{\widehat{\eta}_1^{\widehat{\beta}_1+1}\widehat{\eta}_2^{\widehat{\beta}_2+1}}e^{-(x_1/\widehat{\eta}_1)^{\widehat{\beta}_1}-(x_2/\widehat{\eta}_2)^{\widehat{\beta}_2}}}{\big[1+\theta\big(2e^{-(x_1/\widehat{\eta}_1)^{\widehat{\beta}_1}}-1\big)\big(2e^{-(x_2/\widehat{\eta}_2)^{\widehat{\beta}_2}}-1\big)\big]^2},\\
M_{44}=&-\frac{\widehat{\beta}_2}{\widehat{\eta}_2^2}+\frac{\widehat{\beta}_2(\widehat{\beta}_2+1)}{\widehat{\eta}_2^{\widehat{\beta}_2+2}}x_2^{\widehat{\beta}_2}-\frac{A_{44}}{\big[1+\theta\big(2e^{-(x_1/\widehat{\eta}_1)^{\widehat{\beta}_1}}-1\big)\big(2e^{-(x_2/\widehat{\eta}_2)^{\widehat{\beta}_2}}-1\big)\big]^2},\\
A_{44}=&2\theta\widehat{\beta}_2\frac{x_2^{\widehat{\beta}_2}}{\widehat{\eta}_2^{\widehat{\beta}_2+1}}e^{-(x_2/\widehat{\eta}_2)^{\widehat{\beta}_2}}\left(2e^{-(x_1/\widehat{\eta}_1)^{\widehat{\beta}_1}}-1\right)\Big[\frac{1}{\widehat{\eta}_2}\big(\widehat{\beta}_2(\frac{x_2}{\widehat{\eta}_2})^{\widehat{\beta}_2}-(\widehat{\beta}_2+1)\big)\\
&\times\big[1+\theta\big(2e^{-(x_1/\widehat{\eta}_1)^{\widehat{\beta}_1}}-1\big)\big(2e^{-(x_2/\widehat{\eta}_2)^{\widehat{\beta}_2}}-1\big)\big]-\frac{x_2^{\widehat{\beta}_2}}{\widehat{\eta}_2^{\widehat{\beta}_2+1}}e^{-(x_2/\widehat{\eta}_2)^{\widehat{\beta}_2}}\Big].
\end{aligned}$$

Due to the symmetry property of the Fisher information matrix, it holds that $M_{21}=M_{12}$, $M_{31}=M_{13}$, $M_{32}=M_{23}$, $M_{41}=M_{14}$, $M_{42}=M_{24}$ and $M_{43}=M_{34}$.