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Research Paper

A generalized δ-shock model in the discrete-time framework

Reza Farhadian*, Habib Jafari
Department of Statistics, Razi University, Kermanshah, Iran

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Abstract: According to the δ-shock model, a system subjected to random shocks fails when the inter-arrival time between two successive shocks is less than a critical threshold δ. Several generalizations of the δ-shock model have been proposed to study shock-exposed systems in different scenarios. In this paper, we examine a recently proposed generalization of the δ-shock model within a discrete-time framework, where inter-arrival times follow a geometric distribution. We derive the probability generating function, the mean time to failure, and the variance of the system lifetime. Furthermore, we present an illustrative example related to crop insurance is presented to demonstrate the practical relevance and applicability of the proposed discrete-time model.

Keywords: Crop insurance; Discrete-time framework; Generalized δ-shock model.
Mathematics Subject Classification (2010): 62N05, 90B25, 60E05.

1 Introduction

Shock models are useful tools for studying the lifetime behavior of systems that are subjected to random shocks at random times. These models have received considerable attention and have been widely applied in various fields, including applied probability, reliability engineering, economics, and biology. The δ-shock model is one of the shock models that has attracted significant attention in the reliability literature. Under the δ-shock model, system failure occurs when the inter-arrival time between two successive shocks falls below a given critical threshold δ. The δ-shock model was first introduced by Li et al. (1999), and has since been extended in several directions. For example, Eryilmaz (2013) studied a discrete-time version of the δ-shock model in which shocks

*Corresponding author: farhadian.reza@yahoo.com

occur according to a binomial process. Parvardeh and Balakrishnan (2015) introduced a mixed δ-shock model in which the system fails when the inter-arrival time between two successive shocks is smaller than a threshold δ or when the magnitude of the shock exceeds another threshold γ. Wang and Peng (2017) proposed a generalized δ-shock model with two types of shocks and two different recovery times. Specifically, for a type i shock ($i = 1, 2$), the recovery time of the system is δ_i, and the system fails if a shock occurs before the system has recovered from the previous one. Tuncel and Eryilmaz (2018) investigated the survival function and the mean lifetime of the system under the δ-shock model within a proportional hazard rate framework. Poursaeed (2019) introduced a generalized form of the δ-shock model by considering two different critical thresholds and a probable failure region. Poursaeed (2020) studied a discrete-time δ-shock model for a multi-state system in which the system efficiency decreases when the inter-arrival time between two successive shocks falls in the interval $[\delta_1, \delta_2)$ and the system fails when the inter-arrival time is less than δ_1. Lorvand et al. (2020) discussed a mixed δ-shock model for multi-state systems by assuming a renewal shock process with three distinct failure-related states. Lorvand and Eryilmaz (2024) developed a new δ-shock model by considering the shock magnitude, where the system fails if a shock following a non-critical shock occurs within δ_1, or if a shock following a critical shock occurs within δ_2 ($\delta_1 < \delta_2$). Farhadian and Jafari (2025) proposed a generalized version of the δ-shock model in which the critical region for inter-arrival times between shocks is the interval $[\alpha, \delta]$ for $0 \leq \alpha < \delta$. This formulation generalizes the classical δ-shock model by allowing failure to be associated with an interval of inter-arrival times, rather than solely with inter-arrival times shorter than a fixed threshold.

In this paper, we develop a discrete-time version of the generalized δ-shock model proposed by Farhadian and Jafari (2025) using a discrete-time approach. In a discrete-time framework, it is assumed that shocks are delivered to the system at discrete time points, such as days, weeks, or months. We consider a scenario in which the inter-arrival times between shocks follow a geometric distribution. The geometric distribution is a natural discrete-time analogue of the continuous exponential distribution and is particularly suitable for modeling events observed over successive discrete time units. For example, rainfall in a certain geographical area can act as a damaging shock to systems sensitive to rainy weather, and the time intervals between rainfall events may be modeled by a geometric distribution when observations are recorded over discrete time units.

The main contributions of this paper are to formulate a discrete-time version of the generalized δ-shock model, derive the main characteristics of the system lifetime under geometrically distributed inter-arrival times, investigate system reliability through a matrix-geometric approach, and provide an illustrative example with an application to crop insurance. The rest of the paper is organized as follows. Section 2 describes the model and its underlying assumptions. The characteristics of the system lifetime are derived in Section 3. Section 4 examines the system reliability using a matrix-geometric distribution. Section 5 presents an illustrative example with an application to crop insurance. Finally, Section 6 concludes the paper.

2 The model

Consider a system that starts operating at time $t = 0$ and is subjected to a sequence of random external shocks, which are the only cause of system failure. Let X_1 denote the time from $t = 0$ to the first shock, and for $i \geq 2$, let X_i denote the inter-arrival time between the $(i-1)$th and ith shocks. Assume that $X_1, X_2, \ldots$ are independent and identically distributed (i.i.d.) discrete random variables with cumulative distribution function $F(x) = \Pr(X \leq x)$. Suppose that the system's performance in dealing with random shocks is such that the system fails whenever the time to the first shock or any subsequent inter-arrival time falls within the set $\mathcal{C}_{\alpha,\delta} = \{\alpha, \alpha+1, \ldots, \delta\}$, where $1 \leq \alpha < \delta$. Otherwise, the system continues to operate safely. The lifetime of the system is defined as follows

$$T_{\alpha,\delta} = \sum_{i=1}^{N} X_i,$$

where the random variable N (which has support in $\mathbb{N} = \{1, 2, \ldots\}$) is defined as follows:

$$\{N = n\} = \{X_1 \notin \mathcal{C}_{\alpha,\delta}, X_2 \notin \mathcal{C}_{\alpha,\delta}, \ldots, X_{n-1} \notin \mathcal{C}_{\alpha,\delta}, X_n \in \mathcal{C}_{\alpha,\delta}\}.$$

Since $X_1, X_2, \ldots$ are i.i.d., the random variable N follows a geometric distribution with probability mass function (pmf) $\Pr(N = n) = \big(1 - \Pr(X_1 \in \mathcal{C}_{\alpha,\delta})\big)^{n-1} \Pr(X_1 \in \mathcal{C}_{\alpha,\delta})$ for $n \in \mathbb{N}$. Consequently, the mean of N is

$$E(N) = \frac{1}{\Pr(X_1 \in \mathcal{C}_{\alpha,\delta})} = \frac{1}{F(\delta) - F(\alpha) + \Pr(X_1 = \alpha)}. \tag{1}$$

Indeed, the above shock model can be viewed as a discrete-time version of the generalized δ-shock model introduced by Farhadian and Jafari (2025). At present, we assume that X_i, $i \geq 1$, follow a geometric distribution with the mean $\frac{1}{p}$ for $0 < p < 1$. That is,

$$\Pr(X_i = x) = pq^{x-1}, \qquad x = 1, 2, \ldots, \tag{2}$$

where $q = 1 - p$. Therefore, $\Pr(X_1 \in \mathcal{C}_{\alpha,\delta}) = \sum_{x=\alpha}^{\delta} pq^{x-1} = q^{\alpha-1}\big(1 - q^{\delta-\alpha+1}\big)$, and consequently,

$$\Pr(N = n) = \big(1 - q^{\alpha-1}\big(1 - q^{\delta-\alpha+1}\big)\big)^{n-1} q^{\alpha-1}\big(1 - q^{\delta-\alpha+1}\big), \qquad n \in \mathbb{N}.$$

In the subsequent sections, we investigate some distributional properties of the system lifetime.

3 Properties of the system lifetime

In this section, we investigate some distributional properties of the system lifetime under the model described in Section 2. We first derive the probability generating function (pgf) of the system lifetime.

Theorem 3.1. *The pgf of $T_{\alpha,\delta}$ is given by*

$$G_{T_{\alpha,\delta}}(z) = \frac{pq^{\delta+1}z^{\delta+1} - pq^{\alpha}z^{\alpha}}{pq^{\delta+1}z^{\delta+1} - pq^{\alpha}z^{\alpha} + qz - q}.$$

Proof. Under the assumptions of the model, the inter-arrival times $X_1, X_2, \ldots$ are i.i.d geometrically distributed random variables with pmf given in (2). Therefore, the pgf of the system lifetime $T_{\alpha,\delta}$ can be derived as follows

$$\begin{aligned}
G_{T_{\alpha,\delta}}(z) &= E\left(z^{T_{\alpha,\delta}}\right) = E\left(z^{\sum_{i=1}^{N} X_i}\right) = E\left(E\left(z^{\sum_{i=1}^{N} X_i} | N\right)\right) \\
&= \sum_{n=1}^{\infty} E\left(z^{\sum_{i=1}^{N} X_i} I_{\{N=n\}}\right) \\
&= \sum_{n=1}^{\infty} E\left(z^{\sum_{i=1}^{n} X_i} I_{\{X_1 \notin \mathcal{C}_{\alpha,\delta}, X_2 \notin \mathcal{C}_{\alpha,\delta}, \ldots, X_{n-1} \notin \mathcal{C}_{\alpha,\delta}, X_n \in \mathcal{C}_{\alpha,\delta}\}}\right) \\
&= \sum_{n=1}^{\infty} \left(E(z^{X_1} I_{\{X_1 \notin \mathcal{C}_{\alpha,\delta}\}})\right)^{n-1} E(z^{X_1} I_{\{X_1 \in \mathcal{C}_{\alpha,\delta}\}}) \\
&= \frac{E(z^{X_1} I_{\{X_1 \in \mathcal{C}_{\alpha,\delta}\}})}{1 - E(z^{X_1} I_{\{X_1 \notin \mathcal{C}_{\alpha,\delta}\}})} \\
&= \frac{\sum_{x=\alpha}^{\delta} z^x p(1-p)^{x-1}}{1 - \left(\sum_{x=1}^{\alpha-1} z^x p(1-p)^{x-1} + \sum_{x=\delta+1}^{\infty} z^x p(1-p)^{x-1}\right)} \\
&= \frac{\frac{p(1-p)^{\delta+1} z^{\delta+1} - p(1-p)^{\alpha} z^{\alpha}}{(1-p)^2 z - (1-p)}}{1 - \left(\frac{p(1-p)^{\alpha} z^{\alpha} - p(1-p) z}{(1-p)^2 z - (1-p)} + \frac{p(1-p)^{\delta} z^{\delta+1}}{1-(1-p)z}\right)} \\
&= \frac{p(1-p)^{\delta+1} z^{\delta+1} - p(1-p)^{\alpha} z^{\alpha}}{\left((1-p)^2 z - (1-p)\right)\left(1 + \frac{p(1-p)^{\delta} z^{\delta+1}}{(1-p)z-1} - \frac{p(1-p)^{\alpha} z^{\alpha} - p(1-p)z}{(1-p)^2 z - (1-p)}\right)} \\
&= \frac{p(1-p)^{\delta+1} z^{\delta+1} - p(1-p)^{\alpha} z^{\alpha}}{p(1-p)^{\delta+1} z^{\delta+1} - p(1-p)^{\alpha} z^{\alpha} + ((1-p)^2 + p(1-p))z - (1-p)} \\
&= \frac{p(1-p)^{\delta+1} z^{\delta+1} - p(1-p)^{\alpha} z^{\alpha}}{p(1-p)^{\delta+1} z^{\delta+1} - p(1-p)^{\alpha} z^{\alpha} + (1-p)z - (1-p)}.
\end{aligned}$$

This completes the proof of the theorem. □

In the next simple but important theorem, we obtain an explicit expression for the system's mean time to failure (MTTF).

Theorem 3.2. *The MTTF of the system is given by*

$$MTTF = \frac{1}{p(q^{\alpha-1} - q^{\delta})}.$$

Proof. The random variable N is a stopping time for the sequence $X_1, X_2, \ldots$, i.e., the event $\{N = n\}$ is independent of $X_{n+1}, X_{n+2}, \ldots$ for $n \geq 1$. Therefore, by using the well-known Wald's identity, the MTTF of the system can be computed as follows

$$MTTF = E\left(T_{\alpha,\delta}\right) = E\left(\sum_{i=1}^{N} X_i\right) = E(N)E(X_1).$$

Since N follows a geometric distribution with mean $E(N) = \frac{1}{F(\delta)-F(\alpha)+\Pr(X_1=\alpha)}$, and the inter-arrival times X_i $(i \geq 1)$ are assumed to be geometrically distributed with mean $E(X_1) = \frac{1}{p}$, we obtain

$$\begin{aligned} MTTF = \frac{\frac{1}{p}}{F(\delta) - F(\alpha) + \Pr(X_1 = \alpha)} &= \frac{\frac{1}{p}}{(1-p)^{\alpha} - (1-p)^{\delta} + p(1-p)^{\alpha-1}} \\ &= \frac{1}{p\big((1-p)^{\alpha-1} - (1-p)^{\delta}\big)}, \end{aligned}$$

This completes the proof. □

To prove the next theorem, we require the following lemma (see, e.g., Grimmett and Welsh, 2014).

Lemma 3.3. *Let Y be a discrete random variable with pgf $G_Y(z)$. Then*

$$E\big(Y(Y-1)(Y-2)\dots(Y-k+1)\big) = \frac{d^k}{dz^k} G_Y(z)\bigg|_{z=1}.$$

Theorem 3.4. *The second moment of $T_{\alpha,\delta}$ is given by*

$$E(T^2_{\alpha,\delta}) = \frac{2q^2 + (2\delta+1)pq^{\delta+2} - (2\alpha-1)pq^{\alpha+1}}{p^2\big(q^{\delta+1} - q^{\alpha}\big)^2}.$$

Proof. Using Lemma 3.3 for $k = 2$, together with Theorem 3.1, we obtain

$$E(T^2_{\alpha,\delta} - T_{\alpha,\delta}) = \frac{d^2}{dz^2} G_{T_{\alpha,\delta}}(z)\bigg|_{z=1} = \frac{2q\big(q + (\delta+1)pq^{\delta+1} - \alpha p q^{\alpha}\big)}{p^2\big(q^{\delta+1} - q^{\alpha}\big)^2}. \tag{3}$$

Now, by applying Theorem 3.2 (for $E(T_{\alpha,\delta})$) in (3), we obtain

$$\begin{aligned} E(T^2_{\alpha,\delta}) &= \frac{2q\big(q + (\delta+1)pq^{\delta+1} - \alpha p q^{\alpha}\big)}{p^2\big(q^{\delta+1} - q^{\alpha}\big)^2} + E(T_{\alpha,\delta}) \\ &= \frac{2q\big(q + (\delta+1)pq^{\delta+1} - \alpha p q^{\alpha}\big)}{p^2\big(q^{\delta+1} - q^{\alpha}\big)^2} + \frac{1}{p\big(q^{\alpha-1} - q^{\delta}\big)} \\ &= \frac{2q^2 + (2\delta+1)pq^{\delta+2} - (2\alpha-1)pq^{\alpha+1}}{p^2\big(q^{\delta+1} - q^{\alpha}\big)^2}. \end{aligned}$$

This completes the proof. □

Corollary 3.5. *The variance of $T_{\alpha,\delta}$ is given by*

$$Var(T_{\alpha,\delta}) = \frac{q^2 + (2\delta+1)pq^{\delta+2} - (2\alpha-1)pq^{\alpha+1}}{p^2\big(q^{\delta+1} - q^{\alpha}\big)^2}.$$

Proof. Using the definition of variance, $Var(T_{\alpha,\delta}) = E(T^2_{\alpha,\delta}) - E^2(T_{\alpha,\delta})$, along with Theorems 3.2 and 3.4, the desired result follows immediately. □

4 The matrix-geometric distribution approach

In this section, we show that the system lifetime can follow a matrix-geometric distribution. In comparison to the pgf, the matrix-geometric distribution provides explicit closed form expressions for the pmf and the survival function. We recall that a random variable T taking values in $\mathbb{N}$ has a matrix-geometric distribution if its pgf can be written as follows (see, e.g., Chapter 4 from Bladt and Nielsen, 2017)

$$G_T(z) = \frac{c_1 z + \ldots + c_m z^m}{1 + d_1 z + \ldots + d_m z^m}, \tag{4}$$

for some $m \geq 1$ and real constants $c_1, \ldots, c_m$ and $d_1, \ldots, d_m$.

Based on this representation, the pmf and survival function of T are given as follows, respectively

$$\begin{aligned} \Pr(T = t) &= \boldsymbol{\pi} \mathbf{Q}^{t-1} \mathbf{u}', \\ \Pr(T > t) &= \boldsymbol{\pi} \mathbf{Q}^{t} (\mathbf{I} - \mathbf{Q})^{-1} \mathbf{u}', \end{aligned}$$

where $\mathbf{I}$ is the $m \times m$ identity matrix, $\boldsymbol{\pi}$ is a $1 \times m$ row vector, $\mathbf{Q}$ is an $m \times m$ matrix, and $\mathbf{u}'$ is an $m \times 1$ column vector, defined as follows

$$\boldsymbol{\pi} = (1, 0, \ldots, 0), \quad \mathbf{Q} = \begin{pmatrix} -d_1 & 0 & 0 & \cdots & 0 & 1 \\ -d_m & 0 & 0 & \cdots & 0 & 0 \\ -d_{m-1} & 1 & 0 & \cdots & 0 & 0 \\ -d_{m-2} & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -d_2 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \quad \mathbf{u}' = \begin{pmatrix} c_1 \\ c_m \\ c_{m-1} \\ c_{m-2} \\ \vdots \\ c_2 \end{pmatrix}. \tag{5}$$

We now prove the following theorem.

Theorem 4.1. *Under the model described in Section 2, we have* $\Pr(T_{\alpha,\delta} = t) = \boldsymbol{\pi} \boldsymbol{Q}^{t-1} \boldsymbol{u}'$ *and* $\Pr(T_{\alpha,\delta} > t) = \boldsymbol{\pi} \boldsymbol{Q}^{t} \left(\boldsymbol{I} - \boldsymbol{Q}\right)^{-1} \boldsymbol{u}'$*, where* $\boldsymbol{Q}$ *and* $\boldsymbol{u}'$ *are as defined in* (5) *with* $m = \delta + 1$ *and the following elements:*

$$\begin{aligned} &d_1 = -1, \qquad d_\alpha = pq^{\alpha-1}, \qquad d_{\delta+1} = -pq^\delta, \qquad d_i = 0 \;\; \forall i \neq 1, \alpha, \delta+1, \\ &c_\alpha = pq^{\alpha-1}, \qquad c_{\delta+1} = -pq^\delta, \qquad c_i = 0 \;\; \forall i \neq \alpha, \delta+1. \end{aligned}$$

Proof. From Theorem 3.1, we have

$$\begin{aligned} G_{T_{\alpha,\delta}}(z) &= \frac{pq^{\delta+1} z^{\delta+1} - pq^\alpha z^\alpha}{pq^{\delta+1} z^{\delta+1} - pq^\alpha z^\alpha + qz - q} \\ &= \frac{pq^{\delta+1} z^{\delta+1} - pq^\alpha z^\alpha}{-q\big(1 - z + pq^{\alpha-1} z^\alpha - pq^\delta z^{\delta+1}\big)} \\ &= \frac{pq^{\alpha-1} z^\alpha - pq^\delta z^{\delta+1}}{1 - z + pq^{\alpha-1} z^\alpha - pq^\delta z^{\delta+1}}. \end{aligned} \tag{6}$$

Hence, by analogy with (4) and (6), the elements of $\mathbf{Q}$ are given by $d_1 = -1$, $d_\alpha = pq^{\alpha-1}$, $d_{\delta+1} = -pq^\delta$, and $d_i = 0$ for $i \neq 1, \alpha, \delta+1$. Similarly, for the elements of $\mathbf{u}'$, we have $c_\alpha = pq^{\alpha-1}$, $c_{\delta+1} = -pq^\delta$, and $c_i = 0$ for $i \neq \alpha, \delta+1$. Consequently, the system lifetime $T_{\alpha,\delta}$ follows a matrix-geometric distribution of order $m = \delta + 1$. $\square$

5 Example: application in crop insurance

Rainfed agriculture is a farming system that relies primarily on natural rainfall, without the use of supplemental irrigation. Given its inherent water-use efficiency, rainfed farming is a subject of significant strategic importance in agricultural engineering. In recent years, driven by the challenges of climate change, increasing rainfall variability, and growing concerns over water conservation, these systems have garnered substantial research interest. Typical crops cultivated under rainfed conditions include wheat, barley, chickpea, lentil, and canola. Although many rainfed crops exhibit a certain level of tolerance to water stress, prolonged water deficits during critical growth stages can substantially reduce soil-water availability, impair physiological processes, and ultimately decrease crop yield. Accordingly, rainfed agriculture is often associated with substantial production and economic risk. To reduce their exposure to these financial losses, farmers frequently utilize crop insurance as a mechanism to hedge against yield and income losses resulting from severe, unforeseen drought events.

We apply the proposed model to the financial reliability of a rainfed farm. Rainfall events are treated as shocks, and the inter-arrival times between successive events are measured in weeks. We define financial failure as the occurrence of a financial loss that is not fully covered by insurance. Consider the following states during the growing season:
1. A time interval of less than 3 weeks provides the crop with sufficient water for optimal growth, thereby ensuring high-quality yields and economic profitability;
2. A time interval between 3 and 6 weeks causes moderate water stress and may reduce yield sufficiently to result in a financial loss not fully covered by insurance;
3. A time interval longer than 6 weeks is defined as a severe drought event. Although such an event may cause substantial crop damage, we assume that it activates the insurance contract and hence does not lead to an uncompensated financial loss for the farm.

Therefore, the critical interval for the occurrence of financial failure is $\mathcal{C}_{\alpha,\delta} = [3, 6]$. Under this interpretation, $T_{\alpha,\delta}$ denotes the time until the first rainfall-free interval whose duration lies in the critical interval. We assume that the inter-arrival times between successive rainfall events are i.i.d. random variables following a geometric distribution with parameter p. Applying Theorem 3.1 and Theorem 3.2, together with Corollary 3.5, we derive the pmf, survival function, MTTF, and variance of $T_{\alpha,\delta}$. The corresponding numerical results are summarized in Table 1. The table shows that both the survival function and the MTTF increase as the parameter p increases. This pattern is intuitively reasonable, since a larger value of p corresponds to a shorter mean inter-arrival time between successive rainfall events, namely $\frac{1}{p}$. As a result, the probability of a rainfall-free interval falls within the critical range decreases. Consequently, the crop is more likely to receive sufficient water, thereby reducing the farmer's risk of financial loss. Here, MTTF denotes the mean time to financial loss; therefore, a larger MTTF implies a longer expected time before financial loss occurs.

Table 1: Numerical values of the pmf, survival function, MTTF, and variance of $T_{\alpha,\delta}$.

p	MTTF	$Var(T_{\alpha,\delta})$	t	$\Pr(T_{\alpha,\delta} = t)$	$\Pr(T_{\alpha,\delta} > t)$
0.3	8.95212	58.00824	0	0.00000	1.00000
			1	0.00000	1.00000
			2	0.00000	1.00000
			3	0.14700	0.85300
			4	0.14700	0.70600
			5	0.14700	0.55900
			6	0.12539	0.43361
			7	0.06848	0.36513
			8	0.04687	0.31826
			9	0.02844	0.28982
			10	0.02356	0.26626
0.6	10.69033	63.07856	0	0.00000	1.00000
			1	0.00000	1.00000
			2	0.00000	1.00000
			3	0.09600	0.90400
			4	0.09600	0.80800
			5	0.09600	0.71200
			6	0.08678	0.62521
			7	0.07511	0.55010
			8	0.06589	0.48421
			9	0.05756	0.42664
			10	0.05058	0.37606
0.9	111.12222	11792.62631	0	0.00000	1.00000
			1	0.00000	1.00000
			2	0.00000	1.00000
			3	0.00900	0.99100
			4	0.00900	0.98200
			5	0.00900	0.97300
			6	0.00891	0.96409
			7	0.00883	0.95526
			8	0.00875	0.94651
			9	0.00867	0.93784
			10	0.00859	0.92925

6 Conclusions

In this paper, we investigated a discrete-time version of a generalized δ-shock model, originally introduced by Farhadian and Jafari (2025). We assumed that the inter-arrival times are geometrically distributed, then we derived several distributional and reliability characteristics, including the probability generating function, the mean time to failure, and the variance of the system lifetime. The applicability of the proposed model was demonstrated through an illustrative example.

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