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Research Paper

Goodness-of-fit testing for the inverse Rayleigh half-logistic distribution under progressive Type-II censoring: Entropy and pivotal quantity approaches

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Abstract: In this paper, the problem of goodness-of-fit testing for the inverse Rayleigh-half-logistic distribution with a fixed shape parameter is investigated based on progressively Type-II censored samples. Two test statistics are proposed using entropy-based approaches, along with a non-entropy-based test statistic constructed via the pivotal quantity method. The statistical properties of the suggested test procedures are thoroughly examined, and the corresponding critical values are obtained through Monte Carlo simulation. The power of the proposed tests is evaluated under various censoring schemes and against several alternative hypotheses. The results indicate that the proposed tests perform well overall, particularly against alternatives characterized by decreasing hazard rate functions. Finally, a real data set is analyzed to illustrate the applicability of the proposed methods.

Keywords: Entropy; Goodness-of-fit test; Inverse Rayleigh half-logistic distribution; Progressive Type-II censoring.
Mathematics Subject Classification (2010): 62G10, 94A17, 62N01.

1 Introduction

The inverse Rayleigh distribution, originally introduced by Trayer (1964), has attracted considerable attention in reliability engineering and lifetime analysis due to its flexibility in modeling unimodal right-skewed data and non-monotone hazard rate functions.

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In particular, its hazard rate function initially increases and then decreases toward a stable level, making it suitable for various practical reliability applications. Early inferential properties of this distribution were investigated by Voda (1972), who studied the maximum likelihood estimation of the scale parameter, while Gharraph (1993) derived several important distributional characteristics including the mean, harmonic mean, geometric mean, mode, and median.

Motivated by the increasing development of generator-based lifetime distributions and their successful applications in reliability modeling, several extensions of the inverse Rayleigh model have been proposed in recent years using different generator mechanisms and transformation techniques. Although several inferential and goodness-of-fit procedures have been developed for inverse Rayleigh type models under progressive censoring, extensions specifically tailored to the inverse Rayleigh half-logistic (IRHL) distribution remain limited.

The half-logistic generated family proposed by Cordeiro et al. (2016) provides an efficient mechanism for constructing flexible continuous distributions from a given baseline model. For a baseline cumulative distribution function $G(x;\xi)$ with corresponding density function $g(x;\xi)$, the cumulative distribution function (CDF) and probability density function (PDF) of the half-logistic generated family, denoted by HL-G, are respectively defined as

$$\begin{aligned} F(x;\lambda,\boldsymbol{\xi}) &= \frac{1-\{1-G(x;\boldsymbol{\xi})\}^{\lambda}}{1+\{1-G(x;\boldsymbol{\xi})\}^{\lambda}}, \\ f(x;\lambda,\boldsymbol{\xi}) &= \frac{2\lambda g(x;\xi)\{1-G(x;\boldsymbol{\xi})\}^{\lambda-1}}{\left[1+\{1-G(x;\boldsymbol{\xi})\}^{\lambda}\right]^{2}}, \end{aligned}$$

where $\lambda > 0$ is a shape parameter and $\boldsymbol{\xi} = (\xi_1,\ldots,\xi_k)$ denotes the vector of baseline parameters.

This generator has been extensively employed in the literature to construct flexible lifetime distributions. For example, Anwar and Zahoor (2018) introduced the half-logistic Lomax distribution, whereas Aldahlan (2019) investigated several estimation procedures for this model. Subsequently, Anwar and Bibi (2018) proposed the generalized half-logistic Weibull distribution. Furthermore, Smadi and Alrefaei (2021) developed new extensions of the Rayleigh distribution based on inverted Weibull and Weibull generators. More recently, Almarashi et al. (2020) introduced the IRHL distribution as a flexible extension of the inverse Rayleigh model.

Let X be a random variable following the IRHL distribution with scale parameter α and shape parameter λ. Then, for $\alpha,\lambda > 0$, the CDF and PDF of X are respectively given by

$$\begin{aligned} F_X(x;\alpha,\lambda) &= \frac{1-\left[1-\exp\left(-\left(\frac{\alpha}{x}\right)^2\right)\right]^{\lambda}}{1+\left[1-\exp\left(-\left(\frac{\alpha}{x}\right)^2\right)\right]^{\lambda}}, \\ f_X(x;\alpha,\lambda) &= \frac{4\lambda\alpha^2\exp\left(-\left(\frac{\alpha}{x}\right)^2\right)\left[1-\exp\left(-\left(\frac{\alpha}{x}\right)^2\right)\right]^{\lambda-1}}{x^3\left\{1+\left[1-\exp\left(-\left(\frac{\alpha}{x}\right)^2\right)\right]^{\lambda}\right\}^2}. \end{aligned}$$

The corresponding shapes of the PDF and CDF of the IRHL distribution are presented in Figures 1 and 2 for selected values of α and λ.

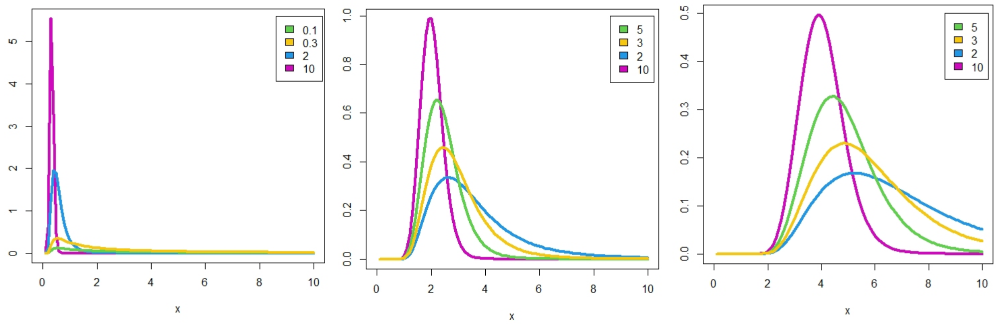

Figure 1: PDF of the IRHL distribution for $\alpha = 0.5$, $\alpha = 3$, and $\alpha = 6$ (from left to right).

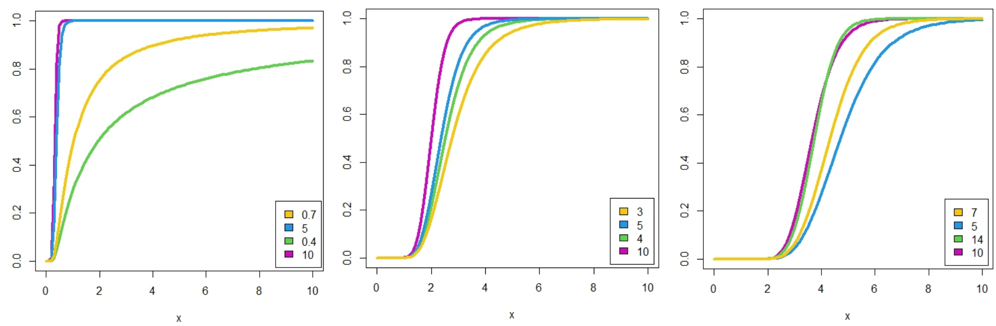

Figure 2: CDF of the IRHL distribution for $\alpha = 0.5$, $\alpha = 3$, and $\alpha = 6$ (from left to right).

In recent years, entropy-based goodness-of-fit procedures have become increasingly important in statistical inference and reliability analysis. Among the most fundamental divergence measures, the Kullback-Leibler (KL) information measure provides an effective criterion for quantifying discrepancies between probability distributions. For two random variables X and Y with density functions $f(\cdot)$ and $g(\cdot)$, respectively, the KL divergence is defined by

$$KL(f:g) = \int f(x) \log \frac{f(x)}{g(x)}\, dx.$$

Several cumulative extensions of the KL divergence have subsequently been proposed, particularly those based on cumulative distribution and survival functions, since they can be conveniently formulated using empirical distribution functions. In this context, Baratpour and Rad (2012) introduced the cumulative residual Kullback–Leibler divergence (CRKL), defined by

$$CRKL(F:G) = \int_0^\infty \bar{F}(x) \log \left(\frac{\bar{F}(x)}{\bar{G}(x)} \right) dx - \big(\mathbb{E}(X) - \mathbb{E}(Y) \big), \tag{1}$$

where $\bar{F}(x) = 1 - F(x)$ and $\bar{G}(x) = 1 - G(x)$ denote the corresponding survival functions. It is well known that $CRKL(F:G) \geq 0$, with equality if and only if $F = G$

almost everywhere. Similarly, Park et al. (2012) proposed the cumulative Kullback-Leibler divergence (CKL), given by

$$CKL(F:G) = \int_0^\infty F(x)\log\left(\frac{F(x)}{G(x)}\right)dx - \big(\mathbb{E}(Y) - \mathbb{E}(X)\big),$$

with the equivalent representation

$$CKL(F:G) = \int_0^\infty F(x)\left(\frac{G(x)}{F(x)} - \log\frac{G(x)}{F(x)} - 1\right)dx.$$

In practical reliability experiments, complete lifetime observations are often unavailable because of time limitations, experimental costs, or other operational restrictions. Consequently, censored sampling schemes naturally arise in survival and reliability studies. Among different censoring mechanisms, progressive Type-II censoring has received substantial attention due to its flexibility and practical applicability.

Suppose that n experimental units are placed in a life-testing experiment and the test terminates at the m-th observed failure ($m < n$). Under progressive Type-II censoring, at the time of the first failure, R_1 surviving units are randomly removed from the remaining $n-1$ units, and at the second observed failure, R_2 surviving units are removed. This process continues until the m-th failure occurs, where the remaining number of censored units is given by $R_m = n - m - \sum_{i=1}^{m-1} R_i$. The resulting progressively censored sample is denoted by $x_{1:m:n} < x_{2:m:n} < \cdots < x_{m:m:n}$ with censoring scheme $\boldsymbol{R} = (R_1, \ldots, R_m)$. Conventional Type-II censoring is obtained as a special case when $R_1 = \cdots = R_{m-1} = 0$ and $R_m = n - m$, while the complete sample case corresponds to $R_1 = \cdots = R_m = 0$.

Recently, goodness-of-fit procedures for Rayleigh-type lifetime models under progressive censoring have continued to receive growing attention in the statistical literature. For example, Kong et al. (2025) proposed goodness-of-fit tests for the two-parameter Rayleigh distribution based on cumulative hazard functions under progressive Type-II censoring. Although the IRHL distribution and related generalized inverse Rayleigh models have recently gained increasing attention in reliability and survival analysis due to their flexibility in modeling complex hazard rate behaviors (see Almarashi et al., 2020; Smadi and Alrefaei, 2021), the goodness-of-fit assessment for such models under progressive Type-II censored samples remains largely unexplored in the literature. At the same time, entropy-based divergence measures, particularly the cumulative residual Kullback-Leibler (CRKL) and cumulative Kullback-Leibler (CKL) information measures, have been shown to be effective tools in statistical inference and goodness-of-fit tests because of their attractive theoretical and computational properties see (see Rao et al., 2004). Nevertheless, their application to progressively censored goodness-of-fit problems for the IRHL distribution has received little attention so far.

Motivated by these observations, the present study develops new goodness-of-fit tests for the IRHL distribution under progressively Type-II censoring using entropy-related divergence measures together with a competing procedure based on the pivotal quantity approach. In the present study, we consider the fixed-shape setting, where λ is specified and α is estimated. This provides a focused framework for developing goodness-of-fit procedures under progressive Type-II censoring. The remainder of the paper is organized as follows. Section 2 discusses parameter estimation for the IRHL distribution under progressively Type-II censored samples. Sections 3 through 5 develop the proposed goodness-of-fit test statistics based on the CRKL divergence, CKL divergence, and pivotal quantity approach, respectively. Section 6 presents simulation results, and Section 7 analyzes a real data set.

2 Estimation of the scale parameter

Although the IRHL distribution is generally a two-parameter model with scale parameter α and shape parameter λ, the present study focuses on the fixed-shape setting. Specifically, $\lambda = \lambda_0$ is treated as specified, while the scale parameter α is unknown and estimated from the progressively Type-II censored sample. This setting defines the scope of the proposed goodness-of-fit procedures. In this section, we derive the maximum likelihood estimator (MLE) of the scale parameter α of the IRHL distribution with fixed shape parameter λ, based on progressively Type-II censored samples. The likelihood function is given by

$$L(\alpha) = C \prod_{i=1}^{m} f(x_i, \alpha, \lambda_0) \left\{1 - F(x_i, \alpha, \lambda_0)\right\}^{R_i},$$

where $C = n(n-R_1-1)(n-R_1-R_2-2)\cdots(n-m+1-\sum_{i=1}^{m-1} R_i)$ is the normalizing constant.

After substituting the expressions for the PDF and CDF and taking the natural logarithm, the log-likelihood function becomes

$$\begin{aligned} \log L(\alpha) \quad &= \quad \sum_{i=1}^{m} \log \left(\frac{4\lambda_0 \alpha^2 \exp\left(-(\alpha/x_i)^2\right) \left[1 - \exp\left(-(\alpha/x_i)^2\right)\right]^{\lambda_0 - 1}}{x_i^3 \left\{1 + \left[1 - \exp\left(-(\alpha/x_i)^2\right)\right]^{\lambda_0}\right\}^2} \right) \\ &\quad + \sum_{i=1}^{m} R_i \log \left(\frac{2\left[1 - \exp\left(-(\alpha/x_i)^2\right)\right]^{\lambda_0}}{1 + \left[1 - \exp\left(-(\alpha/x_i)^2\right)\right]^{\lambda_0}} \right). \qquad (2) \end{aligned}$$

Due to the complexity of the log-likelihood equation, the MLE of α cannot be obtained in closed form. Therefore, we employ the `optimise` function in R to numerically maximize the log-likelihood and obtain the estimate of α.

3 Goodness-of-fit test based on cumulative residual Kullback-Leibler

Let $x_{1:m:n} < x_{2:m:n} < \ldots < x_{m:m:n}$ be a progressively Type-II censored sample with censoring scheme $\boldsymbol{R} = (R_1, \ldots, R_m)$. For simplicity, we denote the ordered observations by $x_1 < x_2 < \ldots < x_m$. In this study, the goodness-of-fit problem is considered under the fixed-shape specification

$$H_0 : F(x) = F_0(x; \alpha, \lambda_0) \quad \alpha > 0,$$

where λ_0 is a specified shape parameter and α is unknown.

The alternative hypothesis is:

$$H_1 : F(x) \neq F_0(x; \alpha, \lambda_0),$$

where

$$F_0(x; \alpha, \lambda_0) = \frac{1 - \left\{1 - \exp\left(-(\alpha/x)^2\right)\right\}^{\lambda_0}}{1 + \left\{1 - \exp\left(-(\alpha/x)^2\right)\right\}^{\lambda_0}}, \quad \alpha, \lambda_0 > 0,$$

is the CDF of the inverse Rayleigh half-logistic distribution.

Following Ma and Gui (2021), the empirical distribution function for progressively Type-II censored samples is given by

$$F_{m:n}(x) = \begin{cases} 0, & x < x_1, \\ p_i, & x_i \leq x < x_{i+1}, \quad i = 1, 2, \ldots, m-1, \\ p_m, & x \geq x_m, \end{cases} \tag{3}$$

where $p_i = \mathbb{E}(U_i)$ and U_i's are the order statistics from a Uniform$(0, 1)$ distribution under the same censoring scheme.

Since the CRKL divergence is nonnegative and equals zero if and only if the two distributions coincide, it provides a natural measure of discrepancy between the empirical distribution $F_{m:n}(x)$ and the hypothesized distribution $F_0(x; \alpha, \lambda_0)$. Therefore, larger values of $CRKL(F_{m:n} : F_0)$ indicate a greater departure from H_0, making it suitable as a goodness-of-fit test statistic. Using the cumulative residual Kullback–Leibler divergence in (1) and the empirical estimator $F_{m:n}(x)$ in (3), the proposed test statistic is given by

$$\begin{aligned} CRKL(F_{m:n} : F_0) &= \int_0^{x_m} \big(1 - F_{m:n}(x)\big) \log\left(\frac{1 - F_{m:n}(x)}{1 - F_0(x; \alpha, \lambda_0)}\right) \mathrm{d}x \\ &\quad - \left[\int_0^{x_m} \big(1 - F_{m:n}(x)\big)\mathrm{d}x - \int_0^{x_m} \big(1 - F_0(x; \alpha, \lambda_0)\big)\, \mathrm{d}x\right]. \end{aligned} \tag{4}$$

Under the null hypothesis,

$$1 - F_0(x; \alpha, \lambda_0) = \frac{2\left(1 - e^{-(\alpha/x)^2}\right)^{\lambda_0}}{1 + \left(1 - e^{-(\alpha/x)^2}\right)^{\lambda_0}}.$$

Substituting the expression for $1 - F_0(x; \alpha, \lambda_0)$ under H_0 into (4) and simplifying yields

$$\begin{aligned} CRKL(F_{m:n} : F_0) &= \sum_{i=0}^{m-1} (1 - p_i) \log(1 - p_i)(x_{i+1} - x_i) \\ &\quad - \sum_{i=0}^{m-1} (1 - p_i) \int_{x_i}^{x_{i+1}} \log\left[2\left(1 - e^{-(\alpha/x)^2}\right)^{\lambda_0}\right] \mathrm{d}x \\ &\quad + \sum_{i=0}^{m-1} (1 - p_i) \int_{x_i}^{x_{i+1}} \log\left[1 + \left(1 - e^{-(\alpha/x)^2}\right)^{\lambda_0}\right] \mathrm{d}x \end{aligned}$$

$$-\sum_{i=0}^{m-1}(1-p_i)(x_{i+1}-x_i)+\int_0^{x_m}\frac{2\left(1-e^{-(\alpha/x)^2}\right)^{\lambda_0}}{1+\left(1-e^{-(\alpha/x)^2}\right)^{\lambda_0}}\,\mathrm{d}x.$$

Replacing α by its maximum likelihood estimator $\hat{\alpha}$ obtained from (2), and standardizing the statistic by

$$\sum_{i=0}^{m-1}(1-p_i)(x_{i+1}-x_i),$$

the final test statistic is defined as

$$T_{CRKL}=Q-W+E+R-1, \tag{5}$$

where

$$\begin{aligned}
Q &= \frac{\sum_{i=0}^{m-1}(1-p_i)\log(1-p_i)(x_{i+1}-x_i)}{\sum_{i=0}^{m-1}(1-p_i)(x_{i+1}-x_i)},\\
W &= \frac{\sum_{i=0}^{m-1}(1-p_i)\int_{x_i}^{x_{i+1}}\log\left[2\left(1-e^{-(\hat{\alpha}/x)^2}\right)^{\lambda_0}\right]\mathrm{d}x}{\sum_{i=0}^{m-1}(1-p_i)(x_{i+1}-x_i)},\\
E &= \frac{\sum_{i=0}^{m-1}(1-p_i)\int_{x_i}^{x_{i+1}}\log\left[1+\left(1-e^{-(\hat{\alpha}/x)^2}\right)^{\lambda_0}\right]\mathrm{d}x}{\sum_{i=0}^{m-1}(1-p_i)(x_{i+1}-x_i)},\\
R &= \frac{\int_0^{x_m}\frac{2\left(1-e^{-(\hat{\alpha}/x)^2}\right)^{\lambda_0}}{1+\left(1-e^{-(\hat{\alpha}/x)^2}\right)^{\lambda_0}}\,\mathrm{d}x}{\sum_{i=0}^{m-1}(1-p_i)(x_{i+1}-x_i)}.
\end{aligned}$$

4 Goodness-of-fit test based on cumulative Kullback-Leibler

Under the same fixed-shape null hypothesis, using the cumulative Kullback-Leibler divergence and the empirical distribution function $F_{m:n}(x)$ given in (3), the test statistic is constructed as follows:

$$\begin{aligned}
CKL(F_{m:n}:F_0) &= \int_0^{x_m}F_{m:n}(x)\log\left(\frac{F_{m:n}(x)}{F_0(x;\alpha,\lambda_0)}\right)\mathrm{d}x-\big(\mathbb{E}(Y)-\mathbb{E}(X)\big)\\
&= \int_0^{x_m}F_{m:n}(x)\log\left(\frac{F_{m:n}(x)}{F_0(x;\alpha,\lambda_0)}\right)\mathrm{d}x\\
&\quad-\left[\int_0^{x_m}\bar{F}_0(x;\alpha,\lambda_0)\,\mathrm{d}x-\int_0^{x_m}\big(1-F_{m:n}(x)\big)\,\mathrm{d}x\right],
\end{aligned} \tag{6}$$

where $\bar{F}_0(x;\alpha,\lambda_0)=1-F_0(x;\alpha,\lambda_0)$. This expression can be decomposed as

$$A = \int_0^{x_m}F_{m:n}(x)\log F_{m:n}(x)\,\mathrm{d}x=\sum_{i=0}^{m-1}p_i\log p_i\,(x_{i+1}-x_i),$$

$$
\begin{aligned}
B &= \int_0^{x_m} F_{m:n}(x) \log\left(\frac{1-\left\{1-\exp\left(-(\alpha/x)^2\right)\right\}^{\lambda_0}}{1+\left\{1-\exp\left(-(\alpha/x)^2\right)\right\}^{\lambda_0}}\right) \mathrm{d}x, \\
C &= \int_0^{x_m} \big(1-F_{m:n}(x)\big)\,\mathrm{d}x = \sum_{i=0}^{m-1}(1-p_i)(x_{i+1}-x_i).
\end{aligned}
$$

Substituting the expressions for A, B, and C into (6), and replacing the unknown parameter α with its maximum likelihood estimator $\hat{\alpha}$ (obtained from equation (2)) while keeping λ fixed, yields the test statistic. To make the statistic scale-free, we normalize it by dividing by $\sum_{i=0}^{m-1}(1-p_i)(x_{i+1}-x_i)$. The final form of the proposed test statistic is

$$
T_{CKL} = F - G + S - K + 1, \tag{7}
$$

where

$$
\begin{aligned}
F &= \frac{\sum_{i=0}^{m-1} p_i \log p_i \,(x_{i+1}-x_i)}{\sum_{i=0}^{m-1}(1-p_i)(x_{i+1}-x_i)}, \\
G &= \frac{\sum_{i=0}^{m-1} p_i \int_{x_i}^{x_{i+1}} \log\Big(1-\big\{1-\exp(-(\hat{\alpha}/x)^2)\big\}^{\lambda_0}\Big)\,\mathrm{d}x}{\sum_{i=0}^{m-1}(1-p_i)(x_{i+1}-x_i)}, \\
S &= \frac{\sum_{i=0}^{m-1} p_i \int_{x_i}^{x_{i+1}} \log\Big(1+\big\{1-\exp(-(\hat{\alpha}/x)^2)\big\}^{\lambda_0}\Big)\,\mathrm{d}x}{\sum_{i=0}^{m-1}(1-p_i)(x_{i+1}-x_i)}, \\
K &= \frac{\int_0^{x_m} \frac{2\left\{1-\exp(-(\hat{\alpha}/x)^2)\right\}^{\lambda_0}}{1+\left\{1-\exp(-(\hat{\alpha}/x)^2)\right\}^{\lambda_0}}\,\mathrm{d}x}{\sum_{i=0}^{m-1}(1-p_i)(x_{i+1}-x_i)}.
\end{aligned}
$$

5 Non-entropy-based test statistic

In this section, a non-entropy-based goodness-of-fit test statistic is proposed using the pivotal quantity approach for the IRHL distribution with a fixed shape parameter. Let $X_{1:m:n}, \ldots, X_{m:m:n}$ be progressively Type-II censored random variables from the IRHL distribution under the censoring scheme $(R_1, \ldots, R_m)$. For notational simplicity, their observed values are denoted by $x_1, \ldots, x_m$. Define

$$
Y_i = -\log\big(1 - F(X_i; \alpha, \lambda_0)\big), \qquad i = 1, 2, \ldots, m,
$$

where

$$
F(X_i; \alpha, \lambda_0) = \frac{1-\left\{1-e^{-(\frac{\alpha}{X_i})^2}\right\}^{\lambda_0}}{1+\left\{1-e^{-(\frac{\alpha}{X_i})^2}\right\}^{\lambda_0}}.
$$

By the probability integral transformation, the Y_i's constitute a progressively Type-II censored sample from the standard exponential distribution. To construct the pivotal quantity, consider the transformations

$$\begin{aligned} S_1 &= nY_1, \\ S_i &= \left[n - \sum_{j=1}^{i-1}(R_j+1)\right](Y_i - Y_{i-1}), \qquad i = 2, \ldots, m. \end{aligned}$$

Following the standard exponential-spacing representation for progressively Type-II censored samples (Balakrishnan and Aggarwala (2000)), the random variables $S_1, S_2, \ldots, S_m$ are independent and identically distributed as standard exponential random variables. Moreover, by summing the above transformations and using $\sum_{i=1}^m R_i = n - m$, we obtain

$$\sum_{i=1}^m S_i = \sum_{i=1}^m (R_i+1)Y_i.$$

Hence, the following pivotal quantity can be constructed:

$$\begin{aligned} W(\alpha) &= 2\sum_{i=1}^m S_i \\ &= 2\sum_{i=1}^m (R_i+1)Y_i \sim \chi^2_{2m}. \end{aligned}$$

Since α is unknown, it is replaced by its maximum likelihood estimate $\hat{\alpha}$ in the proposed test statistic. Thus,

$$W = 2\sum_{i=1}^m (R_i+1)\left[-\log\left(\frac{2\left\{1-e^{-(\frac{\hat{\alpha}}{x_i})^2}\right\}^{\lambda_0}}{1+\left\{1-e^{-(\frac{\hat{\alpha}}{x_i})^2}\right\}^{\lambda_0}}\right)\right]. \tag{8}$$

The finite-sample null distribution of W is therefore approximated using Monte Carlo simulation.

6 Simulation studies

To determine the critical values of the proposed test statistics T_{CRKL} and T_{CKL} (defined in (5) and (7)), as well as the pivotal quantity-based statistic W (defined in (8)), a Monte Carlo simulation study was conducted. Since the exact distributions of these test statistics are unknown, critical values were estimated through extensive simulation.

Using the algorithm of Balakrishnan and Sandhu Balakrishnan and Sandhu (1995), we generated 10,000 progressively Type-II censored samples from the IRHL distribution under various censoring schemes presented in Table 1. The critical values were then obtained as the 90% and 95% empirical quantiles of the test statistics $T_{CRKL,0.90}$, $T_{CRKL,0.95}$, $T_{CKL,0.90}$, $T_{CKL,0.95}$, $W_{0.90}$, and $W_{0.95}$.

The simulation study was conducted under the progressive Type-II censoring schemes reported in Table 1, which cover different censoring patterns and sample sizes. The significance levels considered were $\gamma = 0.10$ and $\gamma = 0.05$. For each simulated

Table 1: Censoring schemes used in the simulation study.

Scheme No.	Censoring scheme $(R_1, \ldots, R_m)$	m	n
(1)	$R_1 = 5$, $R_i = 0$ $(i \neq 1)$		
(2)	$R_5 = 5$, $R_i = 0$ $(i \neq 5)$	5	
(3)	$R_1 = 2$, $R_5 = 3$, $R_i = 0$ $(i \neq 1, 5)$		10
(4)	$R_1 = 3$, $R_i = 0$ $(i \neq 1)$		
(5)	$R_7 = 3$, $R_i = 0$ $(i \neq 7)$	7	
(6)	$R_1 = 1$, $R_7 = 2$, $R_i = 0$ $(i \neq 1, 7)$		
(7)	$R_1 = 10$, $R_i = 0$ $(i \neq 1)$		
(8)	$R_{10} = 10$, $R_i = 0$ $(i \neq 10)$	10	
(9)	$R_1 = 5$, $R_{10} = 5$, $R_i = 0$ $(i \neq 1, 10)$		20
(10)	$R_1 = 8$, $R_i = 0$ $(i \neq 1)$		
(11)	$R_{12} = 8$, $R_i = 0$ $(i \neq 12)$	12	
(12)	$R_1 = 4$, $R_{12} = 4$, $R_i = 0$ $(i \neq 1, 12)$		
(13)	$R_1 = 4$, $R_i = 0$ $(i \neq 1)$		
(14)	$R_{16} = 4$, $R_i = 0$ $(i \neq 16)$	16	20
(15)	$R_1 = 2$, $R_{16} = 2$, $R_i = 0$ $(i \neq 1, 16)$		
(16)	$R_1 = 30$, $R_i = 0$ $(i \neq 1)$		
(17)	$R_{10} = 30$, $R_i = 0$ $(i \neq 10)$	10	
(18)	$R_1 = 15$, $R_{10} = 15$, $R_i = 0$ $(i \neq 1, 10)$		40
(19)	$R_1 = 20$, $R_i = 0$ $(i \neq 1)$		
(20)	$R_{20} = 20$, $R_i = 0$ $(i \neq 20)$	20	
(21)	$R_1 = 10$, $R_{20} = 10$, $R_i = 0$ $(i \neq 1, 20)$		
(22)	$R_1 = 10$, $R_i = 0$ $(i \neq 1)$		
(23)	$R_{30} = 10$, $R_i = 0$ $(i \neq 30)$	30	40
(24)	$R_1 = R_{30} = 5$, $R_i = 0$ $(i \neq 1, 30)$		
(25)	$R_1 = 20$, $R_i = 0$ $(i \neq 1)$		
(26)	$R_{40} = 20$, $R_i = 0$ $(i \neq 40)$	40	
(27)	$R_1 = R_{40} = 10$, $R_i = 0$ $(i \neq 1, 40)$		60
(28)	$R_1 = 10$, $R_i = 0$ $(i \neq 1)$		
(29)	$R_{50} = 10$, $R_i = 0$ $(i \neq 50)$	50	
(30)	$R_1 = R_{50} = 5$, $R_i = 0$ $(i \neq 1, 50)$		

sample, the unknown scale parameter was estimated by maximum likelihood, while the shape parameter was fixed at $\lambda = 4$ in the main simulation study.

To evaluate the power of the proposed tests, several alternative distributions with different hazard rate behaviors were considered:

• **Decreasing hazard rate:** Pareto (Lomax) distributions with shape parameters 3 and 4, denoted $P(1, 3)$ and $P(1, 4)$.

• **Increasing hazard rate:** Chi-square distribution with 4 degrees of freedom (χ^2_4) and Weibull distribution with shape 2 and scale 1, denoted $W(2, 1)$.

• **Non-monotonic hazard rate:** Inverse Rayleigh distribution with scale 3, $INR(3)$, and Log-normal distribution with parameters $(1, 1.2)$, denoted $LN(1, 1.2)$.

Since the power of the test statistics depends on the fixed value of the shape parameter λ, a sensitivity analysis was performed. The results (shown in Figure 3) indicate that the performance of the proposed test statistics becomes relatively stable for $\lambda \geq 4$. Therefore, $\lambda = 4$ was selected as a representative fixed value for the main simulation study.

For alternative distributions with non-monotone hazard rate functions, the results in Table 2 and Figures 4 and 5 indicate that at both significance levels $\gamma = 0.10$

and $\gamma = 0.05$, the statistic W performs better under left censoring schemes, whereas T_{CRKL} shows superior performance under right censoring schemes. Under two-sided censoring schemes, both W and T_{CRKL} outperform T_{CKL}. Moreover, from Scheme 20 onward, the powers of all three statistics become nearly identical and approach one as the sample size increases.

For increasing hazard rate functions, Table 3 and Figures 6 and 7 show that, under left censoring schemes, the statistics W and T_{CKL} perform better than T_{CRKL} at both significance levels. However, under right and two-sided censoring schemes, W and T_{CRKL} provide superior performance.

For decreasing hazard rate functions, the results in Table 4 and Figures 8 and 9 demonstrate that the three proposed statistics perform similarly well with powers close to one at both significance levels and under all censoring schemes.

Overall, the simulation results indicate that the power of all proposed tests increases with the sample size and gradually approaches one.

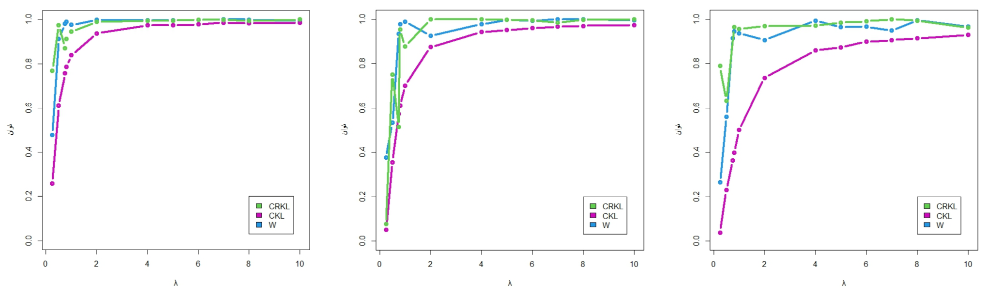

Figure 3: Power of the proposed test statistics for different values of the shape parameter λ. The plots (from left to right) correspond to the alternative distributions Pareto(1,4), Log-normal(1,1.2), and $\chi^2_{(4)}$, respectively, under the progressive Type-II censoring scheme $(0, 0, 0, 0, 5)$.

Figure 4: Power analysis of the proposed test statistics under non-monotonic hazard rate alternative (Inverse Rayleigh distribution with scale parameter 3, $INR(3)$) at significance level $\gamma = 0.10$.

Table 2: Powers of the test statistics T_{CKL}, T_{CRKL}, and W under alternative distributions with non-monotone hazard rate functions.

No.	$INR(3), \gamma = 0.10$			$INR(3), \gamma = 0.05$			$LN(1, 1.2), \gamma = 0.10$			$LN(1, 1.2), \gamma = 0.05$		
	W	T_{CKL}	T_{CRKL}	W	T_{CKL}	T_{CRKL}	W	T_{CKL}	T_{CRKL}	W	T_{CKL}	T_{CRKL}
(1)	0.720	0.707	0.698	0.642	0.624	0.608	1.000	0.987	0.985	1.000	0.981	0.977
(2)	0.464	0.403	0.455	0.365	0.334	0.387	1.000	0.951	0.966	1.000	0.940	0.949
(3)	0.534	0.498	0.559	0.437	0.428	0.456	1.000	0.967	0.974	1.000	0.960	0.965
(4)	0.822	0.795	0.789	0.755	0.731	0.715	1.000	0.996	0.995	1.000	0.994	0.991
(5)	0.671	0.589	0.688	0.570	0.510	0.599	1.000	0.989	0.993	1.000	0.986	0.991
(6)	0.715	0.647	0.731	0.632	0.580	0.651	1.000	0.990	0.995	1.000	0.989	0.992
(7)	0.915	0.881	0.879	0.869	0.831	0.816	1.000	0.999	0.999	1.000	0.999	0.999
(8)	0.628	0.576	0.674	0.519	0.492	0.581	1.000	0.998	0.991	1.000	0.998	0.998
(9)	0.756	0.706	0.794	0.673	0.631	0.717	1.000	0.999	0.999	1.000	0.999	0.999
(10)	0.942	0.917	0.909	0.911	0.874	0.861	1.000	0.999	0.999	1.000	0.999	0.999
(11)	0.754	0.682	0.796	0.656	0.603	0.715	1.000	0.999	0.999	1.000	0.999	0.999
(12)	0.852	0.795	0.874	0.777	0.723	0.819	1.000	0.999	0.999	1.000	0.999	0.999
(13)	0.974	0.956	0.951	0.959	0.933	0.911	1.000	0.999	1.000	1.000	0.999	1.000
(14)	0.926	0.780	0.938	0.885	0.816	0.901	1.000	1.000	1.000	1.000	0.999	1.000
(15)	0.954	0.916	0.952	0.925	0.880	0.925	1.000	0.999	1.000	1.000	0.999	1.000
(16)	0.919	0.793	0.884	0.878	0.846	0.821	1.000	1.000	1.000	1.000	1.000	0.999
(17)	0.417	0.486	0.475	0.311	0.357	0.357	1.000	0.999	0.999	1.000	0.998	0.998
(18)	0.576	0.612	0.630	0.459	0.508	0.541	1.000	0.999	0.999	1.000	0.999	0.999
(19)	0.991	0.977	0.975	0.985	0.963	0.949	1.000	1.000	1.000	1.000	1.000	1.000
(20)	0.815	0.792	0.869	0.725	0.725	0.804	1.000	1.000	1.000	1.000	1.000	1.000
(21)	0.923	0.881	0.945	0.877	0.838	0.910	1.000	1.000	1.000	1.000	1.000	1.000
(22)	0.999	0.997	0.993	0.998	0.994	0.986	1.000	1.000	1.000	1.000	1.000	1.000
(23)	0.986	0.963	0.991	0.970	0.941	0.984	1.000	1.000	1.000	1.000	1.000	1.000
(24)	0.995	0.985	0.992	0.989	0.975	0.988	1.000	1.000	1.000	1.000	1.000	1.000
(25)	0.999	0.999	0.997	0.999	0.999	0.995	1.000	1.000	1.000	1.000	1.000	1.000
(26)	0.990	0.978	0.995	0.979	0.964	0.991	1.000	1.000	1.000	1.000	1.000	1.000
(27)	0.998	0.995	0.999	0.996	0.989	0.997	1.000	1.000	1.000	1.000	1.000	1.000
(28)	0.999	0.999	0.999	0.999	0.999	0.998	1.000	1.000	1.000	1.000	1.000	1.000
(29)	0.999	0.999	0.998	0.999	0.997	0.998	1.000	1.000	1.000	1.000	1.000	1.000
(30)	0.999	0.999	0.998	0.999	0.999	0.998	1.000	1.000	1.000	1.000	1.000	1.000

Figure 5: Power analysis of the proposed test statistics under non-monotonic hazard rate alternative (Inverse Rayleigh distribution with scale parameter 3, $INR(3)$) at significance level $\gamma = 0.05$.

Table 3: Power values of the test statistics T_{CKL}, T_{CRKL}, and W under increasing hazard rate alternative distributions.

No.	$\gamma = 0.10,\ W(2,1)$			$\gamma = 0.05,\ W(2,1)$			$\gamma = 0.10,\ \chi^2_4$			$\gamma = 0.05,\ \chi^2_4$		
	W	T_{CKL}	T_{CRKL}	W	T_{CKL}	T_{CRKL}	W	T_{CKL}	T_{CRKL}	W	T_{CKL}	T_{CRKL}
(1)	0.846	0.841	0.806	0.786	0.782	0.716	0.942	0.940	0.927	0.914	0.912	0.885
(2)	0.830	0.791	0.831	0.778	0.762	0.784	0.901	0.877	0.908	0.864	0.854	0.872
(3)	0.846	0.817	0.852	0.796	0.789	0.794	0.917	0.906	0.923	0.894	0.882	0.890
(4)	0.876	0.868	0.817	0.826	0.824	0.736	0.968	0.961	0.947	0.949	0.945	0.913
(5)	0.889	0.847	0.884	0.847	0.822	0.848	0.958	0.938	0.958	0.935	0.922	0.938
(6)	0.892	0.868	0.887	0.857	0.833	0.832	0.961	0.948	0.959	0.943	0.932	0.939
(7)	0.972	0.961	0.932	0.954	0.950	0.874	0.997	0.996	0.990	0.994	0.992	0.983
(8)	0.969	0.960	0.973	0.950	0.947	0.958	0.990	0.988	0.992	0.985	0.984	0.989
(9)	0.976	0.967	0.977	0.964	0.958	0.962	0.976	0.992	0.996	0.991	0.989	0.993
(10)	0.978	0.968	0.935	0.964	0.956	0.875	0.978	0.996	0.995	0.997	0.995	0.986
(11)	0.979	0.965	0.982	0.969	0.955	0.973	0.979	0.994	0.997	0.993	0.990	0.994
(12)	0.984	0.972	0.984	0.975	0.963	0.971	0.984	0.995	0.998	0.996	0.993	0.995
(13)	0.985	0.978	0.936	0.973	0.969	0.877	0.985	0.998	0.996	0.999	0.998	0.991
(14)	0.989	0.981	0.988	0.984	0.970	0.981	0.989	0.998	0.999	0.998	0.996	0.998
(15)	0.990	0.977	0.982	0.983	0.971	0.970	0.990	0.999	0.998	0.999	0.997	0.997
(16)	0.996	0.993	0.981	0.992	0.988	0.957	0.996	0.999	0.998	0.999	0.999	0.995
(17)	0.984	0.989	0.989	0.977	0.982	0.981	0.984	0.996	0.995	0.990	0.993	0.992
(18)	0.992	0.993	0.993	0.988	0.988	0.991	0.992	0.998	0.998	0.996	0.997	0.998
(19)	0.999	0.997	0.986	0.998	0.994	0.967	0.999	0.999	0.999	1.000	0.999	0.999
(20)	0.998	0.998	0.999	0.998	0.997	0.998	0.998	0.999	1.000	0.999	0.999	0.999
(21)	0.999	0.999	0.999	0.999	0.997	0.999	0.999	0.999	1.000	0.999	0.999	1.000
(22)	0.999	0.999	0.986	0.999	0.999	0.972	0.999	0.999	1.000	1.000	1.000	0.999
(23)	0.999	0.999	0.999	0.999	0.999	0.999	0.999	0.999	1.000	1.000	0.999	0.999
(24)	0.999	0.999	0.999	0.999	0.999	0.999	0.999	1.000	1.000	1.000	1.000	1.000
(25)	1.000	0.999	0.995	0.999	0.999	0.991	1.000	1.000	1.000	1.000	1.000	1.000
(26)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(27)	1.000	1.000	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(28)	1.000	1.000	0.999	0.999	0.955	0.993	1.000	1.000	1.000	1.000	1.000	1.000
(29)	1.000	1.000	1.000	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	1.000
(30)	1.000	1.000	1.000	1.000	0.999	0.999	1.000	1.000	1.000	1.000	1.000	1.000

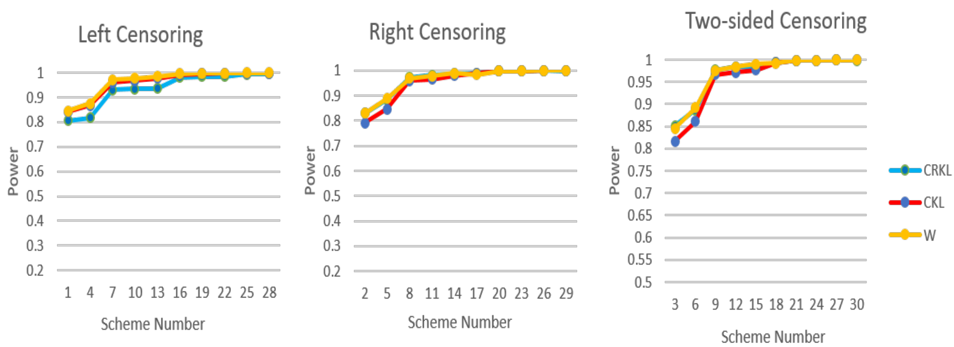

Figure 6: Power analysis of the proposed test statistics under increasing hazard rate alternative (Weibull distribution $W(2,1)$) at significance level $\gamma = 0.10$.

Table 4: Power of the test statistics T_{CKL}, T_{CRKL}, and W for decreasing hazard rate alternatives.

No.	$\gamma = 0.10,\ P(1,4)$			$\gamma = 0.05,\ P(1,4)$			$\gamma = 0.10,\ P(1,3)$			$\gamma = 0.05,\ P(1,3)$		
	W	T_{CKL}	T_{CRKL}	W	T_{CKL}	T_{CRKL}	W	T_{CKL}	T_{CRKL}	W	T_{CKL}	T_{CRKL}
(1)	0.999	0.993	0.991	0.990	0.998	0.986	0.994	0.993	0.993	0.992	0.991	0.988
(2)	0.980	0.974	0.981	0.972	0.970	0.974	0.980	0.978	0.982	0.975	0.970	0.977
(3)	0.987	0.981	0.984	0.981	0.978	0.984	0.987	0.985	0.988	0.982	0.981	0.984
(4)	0.999	0.999	0.997	0.997	0.997	0.994	0.999	0.998	0.998	0.998	0.996	0.996
(5)	0.996	0.994	0.996	0.994	0.991	0.995	0.996	0.994	0.997	0.994	0.994	0.995
(6)	0.998	0.996	0.997	0.995	0.993	0.997	0.997	0.996	0.997	0.996	0.994	0.996
(7)	0.999	1.000	1.000	0.999	1.000	1.000	1.000	0.999	0.999	0.999	1.000	0.999
(8)	0.999	0.999	0.999	0.999	0.999	0.999	0.999	0.999	0.999	0.999	0.999	0.999
(9)	0.999	0.999	1.000	0.999	0.999	1.000	0.999	0.999	1.000	0.999	0.999	0.999
(10)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(11)	0.999	0.999	1.000	0.999	0.999	1.000	0.999	0.999	1.000	0.999	0.999	0.999
(12)	1.000	1.000	1.000	0.999	0.999	1.000	1.000	0.999	1.000	0.999	0.999	0.999
(13)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	1.000	1.000	0.999
(14)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(15)	1.000	1.000	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(16)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

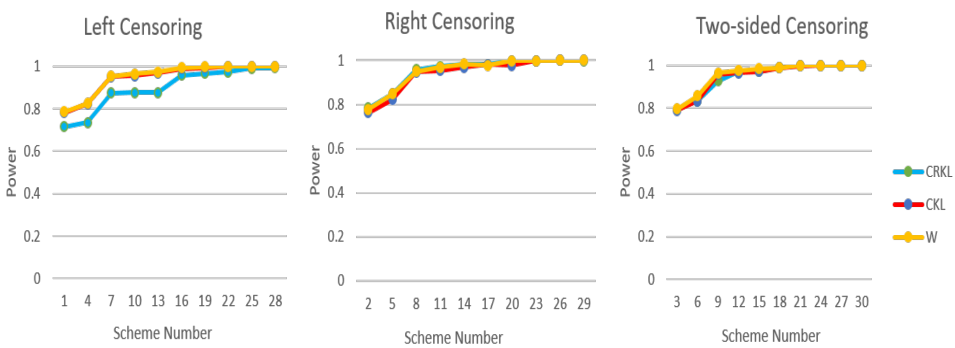

Figure 7: Power analysis of the proposed test statistics under increasing hazard rate alternative (Weibull distribution $W(2,1)$) at significance level $\gamma = 0.05$.

Figure 8: Power analysis of the proposed test statistics under decreasing hazard rate alternative (Pareto distribution $P(1,4)$) at significance level $\gamma = 0.10$.

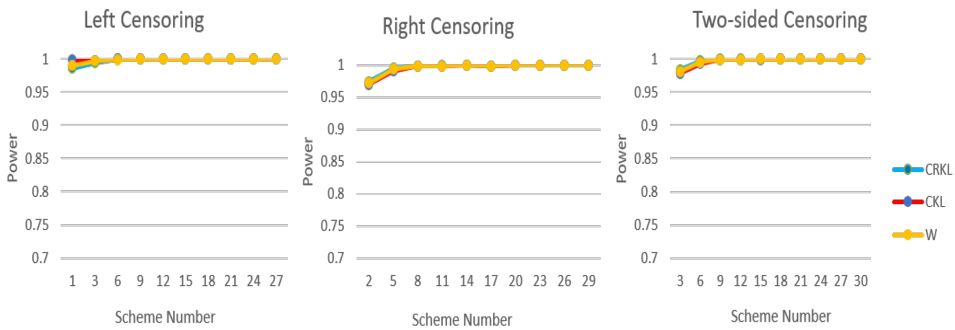

Figure 9: Power analysis of the proposed test statistics under decreasing hazard rate alternative (Pareto distribution $P(1,4)$) at significance level $\gamma = 0.05$.

7 Real data analysis

In this section, a real data set is analyzed using the proposed test statistics T_{CRKL}, T_{CKL}, and W. These data were previously analyzed by Almarashi et al. (2020) for statistical inference based on the IRHL distribution. Moreover, Kundu and Raqab (2009) used the same data set for estimating $R = P(Y < X)$ in the three-parameter Weibull distribution. The data represent the tensile strength (measured in GPA) of single-carbon fibers and 1000-carbon fibers. Using the Kolmogorov-Smirnov (K-S) test, it can be shown that the IRHL distribution provides a good fit to the data. Therefore, a progressively Type-II censored sample is extracted from the data under the censoring scheme $(20, 20 * 0, 8, 17)$. The resulting censored observations are given as:
$1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.006, 2.021$
$2.027, 2.055, 2.063, 2.098, 2.140, 2.179, 2.224, 2.240, 2.253.$

The test statistics $T_{CRKL} = 0.0063$ and $T_{CKL} = 0.0139$ give p-values of 0.2577 and 0.5234, respectively, while $W = 1653.182$ with p-value 0.1844. Since all p-values exceed 0.05, the null hypothesis is not rejected at the conventional significance level. Furthermore, the entropy-based statistics T_{CKL} and T_{CRKL} provide relatively stronger support for the null hypothesis due to their larger p-values.

8 Conclusion

In this paper, the goodness-of-fit testing problem for the IRHL distribution based on progressively Type-II censored data was investigated. Two entropy-based test statistics and one pivotal quantity-based test statistic were proposed. Their performances were then compared through Monte Carlo simulation studies under various alternative hypotheses and different censoring schemes. The simulation results showed that the proposed test statistics have high power in many situations and are capable of effectively distinguishing the IRHL distribution from competing alternatives particularly when the alternative distributions possess decreasing hazard rate functions. Finally, the applicability of the proposed methods was illustrated through the analysis of a real data set. The proposed procedures are developed under a fixed-shape specification. Extending the methodology to the case where both α and λ are unknown, including the derivation of the corresponding null distribution, is an important direction for future research.

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