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Usual stochastic orderings of extreme order statistics with Archimedean copula and new extended Weibull random variables

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Abstract: In this paper, the usual stochastic order is investigated for the minimum and maximum of dependent observations from a new extended Weibull distribution. The observations are coupled using Archimedean copulas. Some results are established for the case where certain shape parameters vary while the others are held fixed, using vector majorization techniques. Since the survival function of this distribution has certain monotonicity and log-concavity properties with respect to two of its parameters, and its distribution function is increasing and log-concave in one of them, sufficient conditions are provided-based on existing results in the literature-for the usual stochastic ordering between sample minima or sample maxima, in terms of the heterogeneity of those two parameter vectors. Finally, several examples are presented to demonstrate the applicability of the main results.

Keywords: Archimedean copula; Extreme order statistics; Majorization; New extended Weibull distribution; Stochastic order.
Mathematics Subject Classification (2010): 60E15, 60K10.

1 Introduction

Series and parallel systems are two basic systems that play prominent roles in various applications in reliability engineering. An n-component system with a series (or parallel) structure fails (or works) if at least one of the components of the system fails (or works). Let $X_1, X_2, \ldots, X_n$ denote the lifetimes of n components that can be used to build an n-component system. If $X_{1:n} \leq \ldots \leq X_{n:n}$ denote the ordered lifetimes of the

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components, then it is known that $X_{1:n}$ and $X_{n:n}$ correspond to the lifetimes of series and parallel systems, respectively.

The reliability properties and underlying stochastic characteristics of series and parallel systems have been widely studied in the literature under a broad range of modeling assumptions. For example, stochastic comparisons of the lifetimes of series and parallel systems, in the case of heterogeneous component lifetimes with exponentiated Weibull (EW) distributions, are considered in Fang and Zhang (2015) and Kundu and Chowdhury (2016) and by Balakrishnan et al. (2014) in the case of heterogeneous components with generalized exponential (GE) distributions. Balakrishnan and Zhao (2013) investigated the hazard rate ordering properties of parallel systems formed from heterogeneous gamma-distributed components. In another line of work, Balakrishnan et al. (2014) examined comparison results for series and parallel configurations whose independent component lifetimes follow the GE distribution. Several stochastic ordering results for parallel systems with EW components were established in Fang and Zhang (2015). Fang et al. (2016b) analyzed different types of stochastic comparisons for series and parallel systems built from independent, heterogeneous, lower-truncated Weibull lifetimes. In Chowdhury and Kundu (2017), sufficient conditions were provided for establishing stochastic orderings of parallel systems with log-Lindley distributed components. Various ordering results for series systems composed of independent and heterogeneous Lomax–exponential components were obtained in Nadeb and Torabi (2018). Moreover, Kayal (2019) studied both series and parallel systems whose components follow independent Kumaraswamy–G distributions and derived several ordering relations between their lifetimes. Ordering results for heterogeneous EW components in both system structures were reported in Barmalzan et al. (2019). Additionally, Kizilaslan (2021) established conditions under which multiple stochastic orderings hold for series and parallel systems with independent heterogeneous Gumbel or truncated Gumbel components.

Most research on stochastic comparisons of series and parallel systems has focused on the situation where all system components are assumed to be independent. In practice, however, components operating within the same technical system are often influenced by shared factors (such as environmental conditions, operational settings, and common stressors) that naturally induce dependence among their lifetimes. Because of this, incorporating dependence into reliability models is both realistic and necessary. There are several approaches for modeling such dependence (see Kotz et al., 2019), among which copula theory has emerged as a widely used and powerful framework; for an extensive treatment of copulas, see Nelsen (2006).

Among the many families of copulas, the Archimedean class has received significant attention due to its analytical tractability and flexibility. This family encompasses commonly used copulas such as the Clayton, Ali–Mikhail–Haq, and Gumbel–Hougaard copulas, and also includes the independence copula as a special case. Consequently, results obtained under the Archimedean framework are quite general and automatically cover the independent setting as a particular instance. Several recent papers have investigated stochastic comparisons of order statistics under Archimedean dependence structures (see Li and Fang, 2015; Fang et al., 2016a; Barmalzan et al., 2020; Shekari et al., 2025, among others).

Müller and Scarsini (2005) investigated Archimedean copulas and their implications for positive dependence, providing important foundations for analyzing dependence beyond independence. In the context of stochastic ordering, Barmalzan et al. (2020) studied dependent components under an Archimedean copula framework for the extended exponential distribution, whereas Nadeb et al. (2021) established general results for the usual stochastic ordering of extreme order statistics from dependent random variables under Archimedean copula dependence. In contrast, Sattari et al. (2023) considered heterogeneous new extended Weibull (NEW) components under the assumption of independence. Unlike the independent framework of Sattari et al. (2023), the present study incorporates dependence among the heterogeneous component lifetimes through an Archimedean copula, which introduces additional analytical complexity into the stochastic comparison of the corresponding systems. Although Barmalzan et al. (2020) and Nadeb et al. (2021) are closely related to our dependence framework, neither study considered the NEW distribution in the setting investigated here. This distinction is important because the hazard rate of the NEW distribution may exhibit an upside-down bathtub-shaped behavior. Moreover, the marginal distribution and survival functions of the NEW model are not monotone with respect to the shape parameter β. Hence, the monotonicity conditions used in some existing stochastic ordering results cannot be directly applied to the present setting, and the relevant log-concavity properties must be verified specifically for the NEW distribution.

More recently, Sahoo and Hazra (2024) studied ordering and ageing properties of sequential order statistics governed by Archimedean copulas. Shrahili (2024) established stochastic ordering results for extreme order statistics from dependent samples with Archimedean copulas, with applications to series and parallel systems. Lu (2024) investigated stochastic comparisons of order statistics from dependent and heterogeneous samples under Archimedean copula dependence. More recently, Shojaee et al. (2024) obtained ordering results for series and parallel systems with dependent and heterogeneous components under Archimedean copula dependence. These recent studies further demonstrate the importance of stochastic comparisons for dependent and heterogeneous systems. However, they do not address the stochastic comparison of series and parallel systems with heterogeneous NEW lifetimes under Archimedean copula dependence. In particular, the non-monotonicity of the marginal distribution and survival functions of the NEW model with respect to β, together with its possible upside-down bathtub-shaped hazard rate, requires a separate analysis of the conditions underlying the stochastic ordering results. This provides further motivation for the present study, in which we investigate the usual stochastic ordering of series and parallel systems whose components follow heterogeneous and dependent NEW distributions, with dependence modeled through an Archimedean copula.

In many real-world applications, lifetime data exhibit hazard rate patterns that are monotonic, bathtub-shaped, or even upside-down bathtub-shaped. To appropriately capture such behaviors, a lifetime distribution must be flexible enough to accommodate these various hazard rate forms. Although the Weibull distribution is one of the most frequently used lifetime models in reliability analysis, it is limited by the fact that its hazard rate cannot generate bathtub or upside-down bathtub shapes. Consequently, the standard Weibull model is not adequate for data displaying these patterns. This limitation has motivated the development of several extensions and generalizations of the Weibull distribution aimed at providing greater flexibility.

Peng and Yan (2014) proposed the NEW distribution and investigated several of its properties, including the behavior of the hazard rate function, the Weibull-type probability plot, and skewness. The cumulative distribution function of a NEW distribution with parameters (α, β, λ), written as NEW(α, β, λ), is given by

$$F(x) = 1 - e^{-\alpha x^{\beta} e^{-\lambda/x}}, \quad x > 0,\, \alpha > 0,\, \beta > 0,\, \lambda \geq 0.$$

If $\lambda = 0$, then the NEW distribution reduces to the Weibull distribution with shape parameter β and scale parameter α. Throughout this paper, it is always assumed that $\lambda > 0$. Further, the parameters β and λ are referred to as the first and the second shape parameters, respectively. The hazard rate function of the NEW distribution admits different forms for different choices of the first shape parameter β. Specifically, for $\beta < 1$, it has an upside-down bathtub shape. For a detailed discussion on this distribution, one may see Peng and Yan (2014).

As mentioned, recently Sattari et al. (2023) obtained ordering results for parallel systems as well as series systems having independent heterogeneous NEW distributed components. By removing the condition of independence, this paper is devoted to further investigating how the heterogeneity of the sample affects order statistics. We consider stochastic comparisons of series and parallel systems with dependent samples having NEW distributed component lifetimes. Sufficient conditions are established for the usual stochastic ordering between the lifetimes of series and parallel systems with dependent components following the NEW model.

The remainder of the paper is organized as follows. Section 2 introduces and presents the required definitions and several useful lemmas that are used throughout the paper. Section 3 studies the usual stochastic ordering of series and parallel systems whose components follow heterogeneous and dependent NEW distributions, with dependence modeled through an Archimedean copula. A simulation study is conducted in Section 4, and the paper is concluded in Section 5.

2 Preliminaries and useful lemmas

There exist several notions under which a random variable X can be considered smaller than a random variable Y. In the usual stochastic ordering case, a random variable X with survival function $\bar{F} = 1 - F$ is stochastically smaller than a random variable Y with survival function $\bar{G} = 1 - G$, denoted by $X \leq_{st} Y$, if $\bar{F}(x) \leq \bar{G}(x)$ for all x. For more details on various kinds of stochastic orders, one may refer to Shaked and Shanthikumar (2007).

For a random vector $\boldsymbol{X} = (X_1, \ldots, X_n)$ with the joint survival function $\bar{F}$, the joint distribution function F, univariate marginal distribution functions $F_1, \ldots, F_n$ and univariate survival functions $\bar{F}_1, \ldots, \bar{F}_n$, if there exists some $\hat{C} : [0,1]^n \longrightarrow [0,1]$ and $C : [0,1]^n \longrightarrow [0,1]$ such that, for all x_i, $i = 1, \ldots, n$,

$$\begin{aligned} \bar{F}(x_1, \ldots, x_n) &= \hat{C}(\bar{F}_1(x_1), \ldots, \bar{F}_n(x_n)), \\ F(x_1, \ldots, x_n) &= C(F_1(x_1), \ldots, F_n(x_n)), \end{aligned}$$

then $\hat{C}$ (or C) is called the survival copula (or copula) of $\boldsymbol{X}$.

A real function ϕ is n-monotone on $(a,b) \subseteq \mathbb{R}$ if $(-1)^{n-2}\phi^{(n-2)}$ is decreasing and convex in (a,b) and $(-1)^k\phi^{(k)}(x) \geq 0$ for all $x \in (a,b), k = 0,1,\ldots,n-2$, where $\phi^{(i)}(\cdot)$ is the ith derivative of $\phi(\cdot)$.

For an n-monotone ($n \geq 2$) function $\phi : [0,+\infty) \longrightarrow [0,1]$ with $\phi(0) = 1$ and $\lim_{x\to+\infty}\phi(x) = 0$, let $\psi = \phi^{-1}$ be the right-continuous inverse of ϕ; then

$$C_\phi(u_1,\ldots,u_n) = \phi(\psi(u_1) + \cdots + \psi(u_n)), \quad \text{for all } u_i \in [0,1], i = 1,\ldots,n,$$

is called an Archimedean copula with generator ϕ. Archimedean copulas cover a wide range of dependence structures, including the independence copula. For more details on Archimedean copulas, readers may refer to Nelsen (2006).

Recall that a function f is said to be convex (concave) if $f(\alpha x + (1-\alpha)y) \leq (\geq)\, \alpha f(x) + (1-\alpha)f(y)$, for all x and y in the domain of f and $0 \leq \alpha \leq 1$. Also, a function f is said to be log-convex (log-concave) if $\log f$ is convex (concave).

Majorization orders provide a powerful framework for proving various inequalities. For the foundational notations and terminology of majorization theory, see Marshall et al. (2011). Let $\boldsymbol{x} = (x_1,\ldots,x_n)$ and $\boldsymbol{y} = (y_1,\ldots,y_n)$ be two real vectors, and let $x_{(1)} \leq \ldots \leq x_{(n)}$ be the increasing arrangement of the components of the vector $\boldsymbol{x}$.

Definition 2.1. *The vector $\boldsymbol{x}$ is said to be*
(i) weakly submajorized by the vector $\boldsymbol{y}$ (denoted by $\boldsymbol{x} \preceq_{\mathrm{w}} \boldsymbol{y}$) if $\sum_{i=j}^n x_{(i)} \leq \sum_{i=j}^n y_{(i)}$ for all $j = 1,\ldots,n$,
(ii) weakly supermajorized by the vector $\boldsymbol{y}$ (denoted by $\boldsymbol{x} \overset{\mathrm{w}}{\preceq} \boldsymbol{y}$) if $\sum_{i=1}^j x_{(i)} \geq \sum_{i=1}^j y_{(i)}$ for all $j = 1,\ldots,n$,
(iii) majorized by the vector $\boldsymbol{y}$ (denoted by $\boldsymbol{x} \overset{\mathrm{m}}{\preceq} \boldsymbol{y}$) if $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$ and $\sum_{i=1}^j x_{(i)} \geq \sum_{i=1}^j y_{(i)}$ for all $j = 1,\ldots,n-1$.

Definition 2.2. *(Bon and Păltănea, 1999) A vector $\boldsymbol{x}$ in $\mathbb{R}_+^n$ is said to be p-larger than another vector $\boldsymbol{y}$ in $\mathbb{R}_+^n$ (denoted by $\boldsymbol{x} \succeq^{\mathrm{p}} \boldsymbol{y}$) if*

$$\prod_{i=1}^j x_{(i)} \leq \prod_{i=1}^j y_{(i)}, \quad \textit{for } j = 1,\ldots,n.$$

Before presenting the main results, we collect several preliminary lemmas that will be used throughout the paper. The following lemma presents the necessary and sufficient conditions for the characterization of Schur-convex and Schur-concave functions.

Lemma 2.3. *(Marshall et al., 2011, Theorem 3.A.4) Suppose $\mathbb{I} \subset \mathbb{R}$ is an open interval and $\Phi : \mathbb{I}^n \longrightarrow \mathbb{R}_+$ is continuously differentiable. Necessary and sufficient conditions for Φ to be Schur-convex (Schur-concave) on $\mathbb{I}^n$ are*
(i) Φ is symmetric on $\mathbb{I}^n$,
(ii) for $i \neq j$ and all $z \in \mathbb{I}^n$,

$$(z_i - z_j)\left(\frac{\partial\Phi(z)}{\partial z_i} - \frac{\partial\Phi(z)}{\partial z_j}\right) \geq (\leq)\, 0,$$

where $\frac{\partial\Phi(z)}{\partial z_i}$ denotes the partial derivative of Φ with respect to its i-th argument.

The following lemma establishes conditions for vector functions to preserve the weak majorization order.

Lemma 2.4. *(Marshall et al., 2011, Theorem 3.A.8) For a function l on $A \in \mathbb{R}^n$, $\boldsymbol{x} \preceq_{\rm w} (\overset{\rm w}{\preceq})\boldsymbol{y}$ implies $l(\boldsymbol{x}) \leq l(\boldsymbol{y})$ if and only if it is increasing (decreasing) and Schur-convex on A.*

The following result is useful for establishing stochastic comparisons between Archimedean copulas. In particular, the super-additivity of the composition of the generators plays an important role in determining the ordering of Archimedean copulas. Such composition-based comparisons are closely related to the general theory of dependence structures and Archimedean copulas; see Joe (1997) for a comprehensive treatment of multivariate dependence and copula constructions. We first recall that a function g is said to be super-additive if $g(u+v) \geq g(u) + g(v)$ for all u and v in its domain.

Lemma 2.5. *(Li and Fang, 2015, Lemma A.1) For two n-dimensional Archimedean copulas $C_{\phi_1}(\boldsymbol{u})$ and $C_{\phi_2}(\boldsymbol{u})$, if $\psi_2 \circ \phi_1$ is super-additive, then $C_{\phi_1}(\boldsymbol{u}) \leq C_{\phi_2}(\boldsymbol{u})$ for all $\boldsymbol{u} \in [0,1]^n$.*

It is worth noting that if we have a function f and another function g, the function $f \circ g(x)$, said as "f of g of x" or "$f \circ g$ of x", is the composition of the two functions.

3 Main results

In this section, we examine the usual stochastic orders of parallel and series systems with dependent heterogeneous NEW components. The random vector $\boldsymbol{X} = (X_1, \ldots, X_n)$ is said to follow the NEW distribution if X_i has the marginal distribution function $F_i(x) = 1 - e^{-\alpha x^{\beta_i} e^{-\lambda/x}}$ for $i = 1, \ldots, n$, and the joint dependence structure is modeled by an Archimedean copula. Specifically, by $\boldsymbol{X} \sim \mathrm{NEW}(\alpha, \boldsymbol{\beta}, \lambda, \phi_1)$ we denote the sample that has the Archimedean copula with generator ϕ_1 and follows a NEW distribution.

In the following theorem, we examine two minimum order statistics constructed from two distinct collections of random variables that share the same parameters α and λ, but differ in their sets of shape parameters $\boldsymbol{\beta}$.

Theorem 3.1. *Suppose that, for $\boldsymbol{X} \sim \mathrm{NEW}(\alpha, \boldsymbol{\beta}, \lambda, \phi_1)$ and $\boldsymbol{X^*} \sim \mathrm{NEW}(\alpha, \boldsymbol{\beta^*}, \lambda, \phi_2)$, ϕ_1 or ϕ_2 is log-convex and $\psi_2 \circ \phi_1$ is super-additive. Then $(\beta_1, \ldots, \beta_n) \overset{\rm m}{\succeq} (\beta_1^*, \ldots, \beta_n^*)$ implies $X_{1:n} \leq_{\rm st} X_{1:n}^*$.*

Proof. The random variables $X_{1:n}$ and $X_{1:n}^*$ have their respective survival functions given by, for $x \geq 0$,

$$\begin{aligned}
\bar{F}_{X_{1:n}}(x) &= P(X_k > x, 1 \leq k \leq n) = \phi_1\big(\sum_{i=1}^{n} \psi_1(e^{-\alpha x^{\beta_i} e^{\frac{-\lambda}{x}}})\big) = J(\alpha, \boldsymbol{\beta}, \lambda, x, \phi_1), \\
\bar{F}_{X_{1:n}^*}(x) &= P(X_k > x, 1 \leq k \leq n) = \phi_2\big(\sum_{i=1}^{n} \psi_2(e^{-\alpha x^{\beta_i^*} e^{\frac{-\lambda}{x}}})\big) = J(\alpha, \boldsymbol{\beta^*}, \lambda, x, \phi_2).
\end{aligned}$$

We prove only the case where ϕ_1 is log-convex, and the other case can be handled similarly. The partial derivatives of $J(\alpha,\boldsymbol{\beta},\lambda,x,\phi_1)$ with respect to β_i are

$$\frac{\partial J(\alpha,\boldsymbol{\beta},\lambda,x,\phi_1)}{\partial\beta_i} = -\alpha x^{\beta_i}\log(x)e^{\frac{-\lambda}{x}}e^{-\alpha x^{\beta_i}e^{\frac{-\lambda}{x}}}\frac{\phi_1'\big(\sum_{i=1}^n\psi_1(e^{-\alpha x^{\beta_i}e^{\frac{-\lambda}{x}}})\big)}{\phi_1'\big(\psi_1(e^{-\alpha x^{\beta_i}e^{\frac{-\lambda}{x}}})\big)},$$
$$\text{for all } x>0.$$

To prove Schur-concavity, it follows from Lemma 2.3 that we have to show that for $i\neq j$,

$$(\beta_i-\beta_j)\Bigg(\frac{\partial J(\alpha,\boldsymbol{\beta},\lambda,x,\phi_1)}{\partial\beta_i}-\frac{\partial J(\alpha,\boldsymbol{\beta},\lambda,x,\phi_1)}{\partial\beta_j}\Bigg)\le 0.$$

For $i\neq j$, the fact that ϕ_1 is decreasing implies

$$\begin{aligned}
&(\beta_i-\beta_j)\Bigg(\frac{\partial J(\alpha,\boldsymbol{\beta},\lambda,x,\phi_1)}{\partial\beta_i}-\frac{\partial J(\alpha,\boldsymbol{\beta},\lambda,x,\phi_1)}{\partial\beta_j}\Bigg)\\
&=-\alpha\log(x)e^{\frac{-\lambda}{x}}\phi_1'\Big(\sum_{i=1}^n\psi_1(e^{-\alpha x^{\beta_i}e^{\frac{-\lambda}{x}}})\Big)(\beta_i-\beta_j)\\
&\quad\times\Bigg(\frac{x^{\beta_i}e^{-\alpha x^{\beta_i}e^{\frac{-\lambda}{x}}}}{\phi_1'\big(\psi_1(e^{-\alpha x^{\beta_i}e^{\frac{-\lambda}{x}}})\big)}-\frac{x^{\beta_j}e^{-\alpha x^{\beta_j}e^{\frac{-\lambda}{x}}}}{\phi_1'\big(\psi_1(e^{-\alpha x^{\beta_j}e^{\frac{-\lambda}{x}}})\big)}\Bigg)\\
&\overset{\text{sgn}}{=}(\beta_i-\beta_j)\log(x)\Bigg(\frac{x^{\beta_i}e^{-\alpha x^{\beta_i}e^{\frac{-\lambda}{x}}}}{\phi_1'\big(\psi_1(e^{-\alpha x^{\beta_i}e^{\frac{-\lambda}{x}}})\big)}-\frac{x^{\beta_j}e^{-\alpha x^{\beta_j}e^{\frac{-\lambda}{x}}}}{\phi_1'\big(\psi_1(e^{-\alpha x^{\beta_j}e^{\frac{-\lambda}{x}}})\big)}\Bigg),
\end{aligned}$$

where $\overset{\text{sgn}}{=}$ means that both sides have the same sign. We now distinguish between the two possible ranges of x, since the sign of $\log(x)$ changes at $x=1$.
(i) Let $x\ge 1$. Then $\log(x)\ge 0$, and x_i^{β} is increasing in β_i. Moreover, we have that $e^{-\alpha x^{\beta_i}e^{\frac{-\lambda}{x}}}$ is a decreasing function of β_i. Also ψ_1 is a decreasing function. As a result, for $\beta_i\ge\beta_j$, we have $\psi_1(e^{-\alpha x^{\beta_i}e^{\frac{-\lambda}{x}}})\ge\psi_1(e^{-\alpha x^{\beta_j}e^{\frac{-\lambda}{x}}})$. Further, the log-convexity of ϕ_1 implies the decreasing property of $\frac{\phi_1}{\phi_1'}$. Then

$$\frac{e^{-\alpha x^{\beta_i}e^{\frac{-\lambda}{x}}}}{\phi_1'\big(\psi_1(e^{-\alpha x^{\beta_i}e^{\frac{-\lambda}{x}}})\big)}=\frac{\phi_1(\psi_1(e^{-\alpha x^{\beta_i}e^{\frac{-\lambda}{x}}}))}{\phi_1'\big(\psi_1(e^{-\alpha x^{\beta_i}e^{\frac{-\lambda}{x}}})\big)}$$

is decreasing in $\beta_i>0$. Also the increasing property of x^{β_i} implies that

$$\frac{x^{\beta_i}\phi_1(\psi_1(e^{-\alpha x^{\beta_i}e^{\frac{-\lambda}{x}}}))}{\phi_1'\big(\psi_1(e^{-\alpha x^{\beta_i}e^{\frac{-\lambda}{x}}})\big)}$$

is decreasing in $\beta_i>0$. So, for $\beta_i\ge\beta_j$,

$$(\beta_i-\beta_j)\log(x)\Bigg(\frac{x^{\beta_i}e^{-\alpha x^{\beta_i}e^{\frac{-\lambda}{x}}}}{\phi_1'\big(\psi_1(e^{-\alpha x^{\beta_i}e^{\frac{-\lambda}{x}}})\big)}-\frac{x^{\beta_j}e^{-\alpha x^{\beta_j}e^{\frac{-\lambda}{x}}}}{\phi_1'\big(\psi_1(e^{-\alpha x^{\beta_j}e^{\frac{-\lambda}{x}}})\big)}\Bigg)\le 0.$$

(ii) Let $0 < x < 1$. In this case, $\log(x) < 0$, and x_i^{β} is decreasing in β_i. Furthermore, $e^{-\alpha x^{\beta_i} e^{\frac{-\lambda}{x}}}$ is an increasing function of β_i. Also ψ_1 is a decreasing function. As a result, for $\beta_i \geq \beta_j$, we have $\psi_1(e^{-\alpha x^{\beta_i} e^{\frac{-\lambda}{x}}}) \leq \psi_1(e^{-\alpha x^{\beta_j} e^{\frac{-\lambda}{x}}})$. Further, the log-convexity of ϕ_1 implies the decreasing property of $\frac{\phi_1}{\phi_1'}$. Then

$$\frac{e^{-\alpha x^{\beta_i} e^{\frac{-\lambda}{x}}}}{\phi_1'\left(\psi_1(e^{-\alpha x^{\beta_i} e^{\frac{-\lambda}{x}}})\right)} = \frac{\phi_1(\psi_1(e^{-\alpha x^{\beta_i} e^{\frac{-\lambda}{x}}}))}{\phi_1'\left(\psi_1(e^{-\alpha x^{\beta_i} e^{\frac{-\lambda}{x}}})\right)}$$

is increasing in $\beta_i > 0$. Also the decreasing property of x^{β_i} implies that

$$\frac{x^{\beta_i}\phi_1(\psi_1(e^{-\alpha x^{\beta_i} e^{\frac{-\lambda}{x}}}))}{\phi_1'\left(\psi_1(e^{-\alpha x^{\beta_i} e^{\frac{-\lambda}{x}}})\right)}$$

is increasing in $\beta_i > 0$. Since $\beta_i - \beta_j \geq 0$, but $\log(x) < 0$, we again obtain

$$(\beta_i - \beta_j)\log(x)\left(\frac{x^{\beta_i} e^{-\alpha x^{\beta_i} e^{\frac{-\lambda}{x}}}}{\phi_1'\left(\psi_1(e^{-\alpha x^{\beta_i} e^{\frac{-\lambda}{x}}})\right)} - \frac{x^{\beta_j} e^{-\alpha x^{\beta_j} e^{\frac{-\lambda}{x}}}}{\phi_1'\left(\psi_1(e^{-\alpha x^{\beta_j} e^{\frac{-\lambda}{x}}})\right)}\right) \leq 0.$$

Then, Schur-concavity of $J(\alpha, \boldsymbol{\beta}, \lambda, x, \phi_1)$ follows from Lemma 2.3. According to Lemma 2.4, $(\beta_1, \ldots, \beta_n) \overset{\mathrm{m}}{\succeq} (\beta_1^*, \ldots, \beta_n^*)$ implies that $J(\alpha, \boldsymbol{\beta}, \lambda, x, \phi_1) \leq J(\alpha, \boldsymbol{\beta^*}, \lambda, x, \phi_1)$. On the other hand, since $\psi_2 \circ \phi_1$ is super-additive by Lemma 2.5, we have that $J(\alpha, \boldsymbol{\beta^*}, \lambda, x, \phi_1) \leq J(\alpha, \boldsymbol{\beta^*}, \lambda, x, \phi_2)$. So, it follows that

$$J(\alpha, \boldsymbol{\beta}, \lambda, x, \phi_1) \leq J(\alpha, \boldsymbol{\beta^*}, \lambda, x, \phi_1) \leq J(\alpha, \boldsymbol{\beta^*}, \lambda, x, \phi_2).$$

That is, $X_{1:n} \leq_{\mathrm{st}} X_{1:n}^*$. □

Remark 3.2. *It is worth noting that the condition "$\psi_2 \circ \phi_1$ is superadditive" in Theorem 3.1 is quite general and can be easily verified for many well-known Archimedean copulas. Suppose that X and X^* exhibit one of the following two dependence structures. (i) Gumbel survival copulas with respective generators*

$$\phi_1(x) = e^{-x^{\frac{1}{\theta_1}}}, \quad \phi_2(x) = e^{-x^{\frac{1}{\theta_2}}}, \quad \theta_2 \geq \theta_1 \geq 1;$$

(ii) Archimedean survival copulas with respective generators

$$\phi_1(x) = (x^{\frac{1}{\theta_1}})^{-1}, \quad \phi_2(x) = (x^{\frac{1}{\theta_2}})^{-1}, \quad \theta_2 \geq \theta_1 \geq 1.$$

It is easy to see that ϕ_i is log-convex for $i = 1, 2$. In view of $\psi_2(\phi_1(0)) = 0$ and the convexity of $\psi_2(\phi_1(x)) = x^{\frac{\theta_2}{\theta_1}}$, we conclude that $\psi_2(\phi_1(x))$ is super-additive by Proposition 21.A.11 in Marshall et al. (2011).

The following example illustrates the result of Theorem 3.1.

Example 3.3. *Consider two two-component series systems, A and B. Let the lifetimes of the components of the first system be denoted by* (X_1, X_2) *and those of the second system be denoted by* (X_1^*, X_2^*). *Assume that the lifetimes follow NEW models. Let* $(\beta_1, \beta_2) = (1, 4), (\beta_1^*, \beta_2^*) = (2, 3), \alpha = 2, \lambda = 3$ *and let* $\phi_1(x) = e^{-x^{\frac{1}{\theta_1}}}$ *be for a Gumbel copula with parameter* $\theta_1 = 2$, *and let* $\phi_2(x) = e^{-x^{\frac{1}{\theta_2}}}$ *be for a Gumbel copula with parameter* $\theta_2 = 4$. *All the conditions of Theorem 3.1 are satisfied. Thus, as an application of Theorem 3.1, we conclude that system B has better performance than system A in the sense of the usual stochastic ordering, as can be seen in Figure 1.*

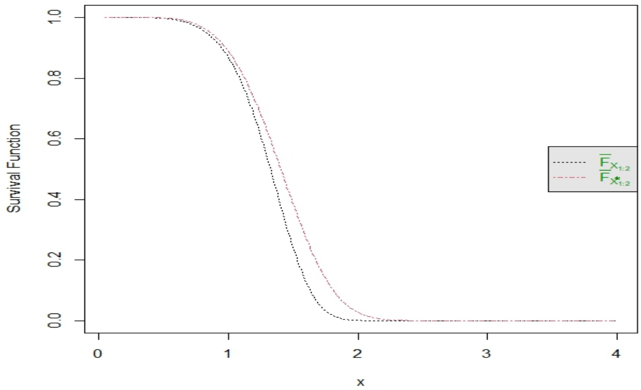

Figure 1: Plot of the survival functions of $X_{1:2}$ and $X_{1:2}^*$.

Naturally, one may wonder whether the sufficient condition in the preceding theorem can be strengthened by replacing weak majorization with the p-larger order. More specifically, since the p-larger order is stronger than weak majorization, it is natural to ask whether

$$(\beta_1, \ldots, \beta_n) \succeq^{\mathrm{p}} (\beta_1^*, \ldots, \beta_n^*)$$

also gives rise to the usual stochastic order between $X_{1:n}$ and $X_{1:n}^*$. The following example gives a negative answer to this conjecture. In other words, although the two parameter vectors satisfy the stronger p-larger order, the corresponding lifetimes need not be ordered in the usual stochastic order. Thus, the sufficient condition based on weak majorization in the preceding theorem cannot, in general, be replaced by the p-larger order.

Example 3.4. *Consider two two-component series systems, A and B. Let the lifetimes of the components of the first system be denoted by* (X_1, X_2) *and those of the second system be denoted by* (X_1^*, X_2^*). *Assume that the lifetimes follow NEW models. Let* $(\beta_1, \beta_2) = (0.1, 0.3), (\beta_1^*, \beta_2^*) = (0.2, 0.3), \alpha = 0.2, \lambda = 1$ *and let* $\phi_1(x) = e^{-x^{\frac{1}{\theta_1}}}$ *be for a Gumbel copula with parameter* $\theta_1 = 2$, *and let* $\phi_2(x) = e^{-x^{\frac{1}{\theta_2}}}$ *be for a Gumbel copula with parameter* $\theta_2 = 4$. *Figure 2 presents a plot of the survival functions of* $X_{1:2}$ *and* $X_{1:2}^*$, *which are seen to cross, and this means that* $X_{1:2} \not\leq_{st} X_{1:2}^*$. *So, the result in*

Theorem 3.1 cannot hold if $(\beta_1, \beta_2) \overset{\mathrm{p}}{\succeq} (\beta_1^*, \beta_2^*)$. *Thus, the parameter vectors satisfy the p-larger order. However, as shown in Figure 2, the corresponding survival functions cross. Consequently, the desired stochastic ordering does not hold. This demonstrates that the sufficient condition based on weak majorization in Theorem 3.1 cannot, in general, be replaced by the stronger p-larger order.*

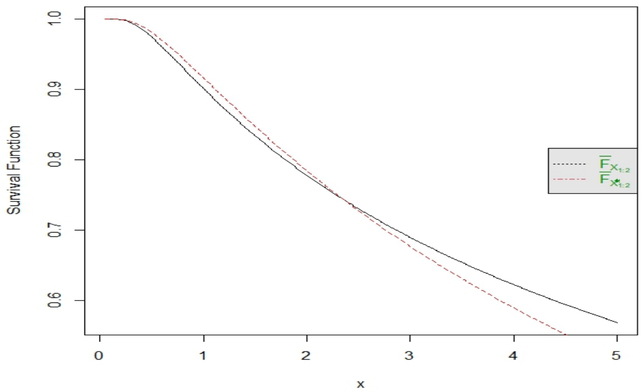

Figure 2: Plot of the survival functions of $X_{1:2}$ and $X^*_{1:2}$.

Nadeb et al. (2021) derived the following theorem for the comparison of series systems under the usual stochastic order.

Theorem 3.5. *Let* $\boldsymbol{X}$ *and* $\boldsymbol{X^*}$ *be two vectors of continuous random variables with* $X_i \sim \bar{F}(x; \lambda_i)$ *and* $X_i^* \sim \bar{F}(x; \lambda_i^*)$, $i = 1, \ldots, n$. *Assume that their associated survival copulas are Archimedean with the respective generators* ϕ_1 *and* ϕ_2. *If* ϕ_1 *or* ϕ_2 *is log-convex,* $\psi_2 \circ \phi_1$ *is super-additive, and for any* x *the function* $\bar{F}(x; \lambda)$ *is decreasing (or increasing) and log-concave in* $\boldsymbol{\lambda}$, *then* $(\lambda_1, \ldots, \lambda_n) \succeq_{\mathrm{w}} [\overset{\mathrm{w}}{\succeq}](\lambda_1^*, \ldots, \lambda_n^*)$ *implies* $X_{1:n} \leq_{\mathrm{st}} X_{1:n}^*$.

It should be noted that our results are different from the results of Nadeb et al. (2021), because $\bar{F}(x, \beta) = e^{-\alpha x^{\beta} e^{-\lambda/x}}$ is not decreasing (or increasing) with respect to β. The following corollary provides a comparison between the smallest order statistics arising from two sets of random variables, when the marginal distributions belong to the NEW distribution.

Corollary 3.6. *Suppose that, for* $\boldsymbol{X} \sim \mathrm{NEW}(\boldsymbol{\alpha}, \beta, \lambda, \phi_1)$ *and* $\boldsymbol{X^*} \sim \mathrm{NEW}(\boldsymbol{\alpha^*}, \beta, \lambda, \phi_2)$, ϕ_1 *or* ϕ_2 *is log-convex and* $\psi_2 \circ \phi_1$ *is super-additive. Then* $(\alpha_1, \ldots, \alpha_n) \succeq_{\mathrm{w}} (\alpha_1^*, \ldots, \alpha_n^*)$ *implies* $X_{1:n} \leq_{\mathrm{st}} X_{1:n}^*$.

Proof. We can see that $\bar{F}(x, \alpha)$ is decreasing and log-concave in α. Hence, Theorem 3.5 immediately implies the result. □

In the following corollary, we compare two minimum order statistics with respect to the usual stochastic order. Here, we assume that two sets of random variables have the same α and β parameters but different λ parameters.

Corollary 3.7. *Suppose that, for $\boldsymbol{X} \sim \mathrm{NEW}(\alpha, \beta, \boldsymbol{\lambda}, \phi_1)$ and $\boldsymbol{X^*} \sim \mathrm{NEW}(\alpha, \beta, \boldsymbol{\lambda^*}, \phi_2)$, ϕ_1 or ϕ_2 is log-convex and $\psi_2 \circ \phi_1$ is super-additive. Then $(\lambda_1, \ldots, \lambda_n) \overset{\mathrm{w}}{\succeq} (\lambda_1^*, \ldots, \lambda_n^*)$ implies $X_{1:n} \leq_{\mathrm{st}} X^*_{1:n}$.*

Proof. We can see that $\bar{F}(x, \lambda)$ is increasing and log-concave in λ. Hence, Theorem 3.5 immediately implies the result. $\square$

In the following theorem, we compare two maximum order statistics arising from random variables that follow the NEW distribution. We show that if the set of shape parameters (β_i's) of one set of random variables is majorized by that of another set, then the maximum order statistic from the first set is dominated by that of the other set with respect to the usual stochastic order.

Theorem 3.8. *Suppose that, for $\boldsymbol{X} \sim \mathrm{NEW}(\alpha, \boldsymbol{\beta}, \lambda, \phi_1)$ and $\boldsymbol{X^*} \sim \mathrm{NEW}(\alpha, \boldsymbol{\beta^*}, \lambda, \phi_2)$, ϕ_1 or ϕ_2 is log-convex and $\psi_2 \circ \phi_1$ is super-additive. Then $(\beta_1, \ldots, \beta_n) \overset{\mathrm{m}}{\succeq} (\beta_1^*, \ldots, \beta_n^*)$ implies $X_{n:n} \geq_{\mathrm{st}} X^*_{n:n}$.*

Proof. The random variables $X_{n:n}$ and $X^*_{n:n}$ have their respective distribution functions given by, for $x \geq 0$,

$$
\begin{aligned}
F_{X_{n:n}}(x) &= \phi_1\big(\sum_{i=1}^{n} \psi_1(1 - e^{-\alpha x^{\beta_i} e^{\frac{-\lambda}{x}}})\big) = \Psi(\alpha, \boldsymbol{\beta}, \lambda, x, \phi_1), \\
F_{X^*_{n:n}}(x) &= \phi_2\big(\sum_{i=1}^{n} \psi_2(1 - e^{-\alpha x^{\beta^*_i} e^{\frac{-\lambda}{x}}})\big) = \Psi(\alpha, \boldsymbol{\beta^*}, \lambda, x, \phi_2).
\end{aligned}
$$

We prove only the case where ϕ_1 is log-convex, and the other case can be handled similarly. The partial derivatives of $\Psi(\alpha, \boldsymbol{\beta}, \lambda, x, \phi_1)$ with respect to β_i are

$$
\begin{aligned}
\frac{\partial \Psi(\alpha, \boldsymbol{\beta}, \lambda, x, \phi_1)}{\partial \beta_i} &= \alpha x^{\beta_i} \log(x) e^{\frac{-\lambda}{x}} e^{-\alpha x^{\beta_i} e^{\frac{-\lambda}{x}}} \frac{\phi_1'\big(\sum_{i=1}^{n} \psi_1(1 - e^{-\alpha x^{\beta_i} e^{\frac{-\lambda}{x}}})\big)}{\phi_1'\big(\psi_1(1 - e^{-\alpha x^{\beta_i} e^{\frac{-\lambda}{x}}})\big)}, \\
&\qquad \text{for all } x > 0.
\end{aligned}
$$

To prove Schur-concavity, it follows from Lemma 2.3 that we have to show that for $i \neq j$,

$$
(\beta_i - \beta_j)\bigg(\frac{\partial \Psi(\alpha, \boldsymbol{\beta}, \lambda, x, \phi_1)}{\partial \beta_i} - \frac{\partial \Psi(\alpha, \boldsymbol{\beta^*}, \lambda, x, \phi_1)}{\partial \beta_j}\bigg) \leq 0.
$$

For $i \neq j$, the fact that ϕ_1 is decreasing implies

$$
\begin{aligned}
&(\beta_i - \beta_j)\bigg(\frac{\partial \Psi(\alpha, \boldsymbol{\beta}, \lambda, x, \phi_1)}{\partial \beta_i} - \frac{\partial \Psi(\alpha, \boldsymbol{\beta^*}, \lambda, x, \phi_1)}{\partial \beta_j}\bigg) \\
&= \alpha \log(x) e^{\frac{-\lambda}{x}} \phi_1'\big(\sum_{i=1}^{n} \psi_1(1 - e^{-\alpha x^{\beta_i} e^{\frac{-\lambda}{x}}})\big)(\beta_i - \beta_j) \\
&\quad \times \bigg(\frac{x^{\beta_i} e^{-\alpha x^{\beta_i} e^{\frac{-\lambda}{x}}}}{\phi_1'\big(\psi_1(1 - e^{-\alpha x^{\beta_i} e^{\frac{-\lambda}{x}}})\big)} - \frac{x^{\beta_j} e^{-\alpha x^{\beta_j} e^{\frac{-\lambda}{x}}}}{\phi_1'\big(\psi_1(1 - e^{-\alpha x^{\beta_j} e^{\frac{-\lambda}{x}}})\big)}\bigg)
\end{aligned}
$$

$$\overset{\text{sgn}}{=}(\beta_i-\beta_j)\log(x)\Bigg(\frac{x^{\beta_j}}{e^{\alpha x^{\beta_j}e^{\frac{-\lambda}{x}}}-1}\times\frac{1-e^{-\alpha x^{\beta_j}e^{\frac{-\lambda}{x}}}}{\phi_1'\big(\psi_1(1-e^{-\alpha x^{\beta_j}e^{\frac{-\lambda}{x}}})\big)}$$
$$-\frac{x^{\beta_i}}{e^{\alpha x^{\beta_i}e^{\frac{-\lambda}{x}}}-1}\times\frac{1-e^{-\alpha x^{\beta_i}e^{\frac{-\lambda}{x}}}}{\phi_1'\big(\psi_1(1-e^{-\alpha x^{\beta_i}e^{\frac{-\lambda}{x}}})\big)}\Bigg).$$

We now distinguish between the two possible ranges of x, since the sign of $\log(x)$ changes at $x=1$.

(i) Let $x\geq 1$. Then, for $\beta_i\geq\beta_j$, we have $\psi_1(1-e^{-\alpha x^{\beta_i}e^{\frac{-\lambda}{x}}})\leq\psi_1(1-e^{-\alpha x^{\beta_j}e^{\frac{-\lambda}{x}}})$. Further, the log-convexity of ϕ_1 implies the decreasing property of $\frac{\phi_1}{\phi_1'}$. Then $\frac{1-e^{-\alpha x^{\beta_i}e^{\frac{-\lambda}{x}}}}{\phi_1'\big(\psi_1(1-e^{-\alpha x^{\beta_i}e^{\frac{-\lambda}{x}}})\big)}=\frac{\phi_1(\psi_1(1-e^{-\alpha x^{\beta_i}e^{\frac{-\lambda}{x}}}))}{\phi_1'\big(\psi_1(1-e^{-\alpha x^{\beta_i}e^{\frac{-\lambda}{x}}})\big)}$ is increasing in $\beta_i>0$. Thus, we have

$$0\leq-\frac{\phi_1(\psi_1(1-e^{-\alpha x^{\beta_i}e^{\frac{-\lambda}{x}}}))}{\phi_1'\big(\psi_1(1-e^{-\alpha x^{\beta_i}e^{\frac{-\lambda}{x}}})\big)}\leq-\frac{\phi_1(\psi_1(1-e^{-\alpha x^{\beta_j}e^{\frac{-\lambda}{x}}}))}{\phi_1'\big(\psi_1(1-e^{-\alpha x^{\beta_j}e^{\frac{-\lambda}{x}}})\big)}.\tag{1}$$

Also the increasing property of x^{β_i} implies that $\frac{x^{\beta_i}}{e^{\alpha x^{\beta_i}e^{\frac{-\lambda}{x}}}-1}$ is decreasing in $\beta_i>0$. So, if $\beta_i\geq\beta_j$; then

$$\frac{x^{\beta_i}}{e^{\alpha x^{\beta_i}e^{\frac{-\lambda}{x}}}-1}\leq\frac{x^{\beta_j}}{e^{\alpha x^{\beta_j}e^{\frac{-\lambda}{x}}}-1}.\tag{2}$$

Thus, using (1) and (2), we obtain

$$\frac{x^{\beta_j}}{e^{\alpha x^{\beta_j}e^{\frac{-\lambda}{x}}}-1}\,\frac{1-e^{-\alpha x^{\beta_j}e^{\frac{-\lambda}{x}}}}{\phi_1'\big(\psi_1(1-e^{-\alpha x^{\beta_j}e^{\frac{-\lambda}{x}}})\big)}\leq\frac{x^{\beta_i}}{e^{\alpha x^{\beta_i}e^{\frac{-\lambda}{x}}}-1}\,\frac{1-e^{-\alpha x^{\beta_i}e^{\frac{-\lambda}{x}}}}{\phi_1'\big(\psi_1(1-e^{-\alpha x^{\beta_i}e^{\frac{-\lambda}{x}}})\big)}.$$

Finally, using the above inequality and the fact $\log(x)\geq 0$ for $x\geq 1$ we get

$$(\beta_i-\beta_j)\bigg(\frac{\partial\Psi(\alpha,\boldsymbol{\beta},\lambda,x,\phi_1)}{\partial\beta_i}-\frac{\partial\Psi(\alpha,\boldsymbol{\beta},\lambda,x,\phi_1)}{\partial\beta_j}\bigg)\leq 0.$$

(ii) Let $0<x\leq 1$. Then, for $\beta_i\geq\beta_j$, we have $\psi_1(1-e^{-\alpha x^{\beta_i}e^{\frac{-\lambda}{x}}})\geq\psi_1(1-e^{-\alpha x^{\beta_j}e^{\frac{-\lambda}{x}}})$. Further, the log-convexity of ϕ_1 implies the decreasing property of $\frac{\phi_1}{\phi_1'}$. Then $\frac{1-e^{-\alpha x^{\beta_i}e^{\frac{-\lambda}{x}}}}{\phi_1'\big(\psi_1(1-e^{-\alpha x^{\beta_i}e^{\frac{-\lambda}{x}}})\big)}=\frac{\phi_1(\psi_1(1-e^{-\alpha x^{\beta_i}e^{\frac{-\lambda}{x}}}))}{\phi_1'\big(\psi_1(1-e^{-\alpha x^{\beta_i}e^{\frac{-\lambda}{x}}})\big)}$ is decreasing in $\beta_i>0$. Thus, we have

$$0\leq-\frac{\phi_1(\psi_1(1-e^{-\alpha x^{\beta_j}e^{\frac{-\lambda}{x}}}))}{\phi_1'\big(\psi_1(1-e^{-\alpha x^{\beta_j}e^{\frac{-\lambda}{x}}})\big)}\leq-\frac{\phi_1(\psi_1(1-e^{-\alpha x^{\beta_i}e^{\frac{-\lambda}{x}}}))}{\phi_1'\big(\psi_1(1-e^{-\alpha x^{\beta_i}e^{\frac{-\lambda}{x}}})\big)}.\tag{3}$$

Also the decreasing property of x^{β_i} implies that $\dfrac{x^{\beta_i}}{e^{\alpha x^{\beta_i} e^{\frac{-\lambda}{x}}}-1}$ is increasing in $\beta_i > 0$. So, if $\beta_i \geq \beta_j$; then

$$\frac{x^{\beta_i}}{e^{\alpha x^{\beta_i} e^{\frac{-\lambda}{x}}}-1} \geq \frac{x^{\beta_j}}{e^{\alpha x^{\beta_j} e^{\frac{-\lambda}{x}}}-1}. \tag{4}$$

Thus, using (3) and (4), we obtain

$$\frac{x^{\beta_j}}{e^{\alpha x^{\beta_j} e^{\frac{-\lambda}{x}}}-1}\,\frac{1-e^{-\alpha x^{\beta_j} e^{\frac{-\lambda}{x}}}}{\phi_1'\big(\psi_1(1-e^{-\alpha x^{\beta_j} e^{\frac{-\lambda}{x}}})\big)} \geq \frac{x^{\beta_i}}{e^{\alpha x^{\beta_i} e^{\frac{-\lambda}{x}}}-1}\,\frac{1-e^{-\alpha x^{\beta_i} e^{\frac{-\lambda}{x}}}}{\phi_1'\big(\psi_1(1-e^{-\alpha x^{\beta_i} e^{\frac{-\lambda}{x}}})\big)}.$$

Finally, using the above inequality and the fact $\log(x) \leq 0$ for $0 < x \leq 1$ we get

$$(\beta_i - \beta_j)\left(\frac{\partial \Psi(\alpha, \boldsymbol{\beta}, \lambda, x, \phi_1)}{\partial \beta_i} - \frac{\partial \Psi(\alpha, \boldsymbol{\beta}, \lambda, x, \phi_1)}{\partial \beta_j}\right) \leq 0.$$

Then the Schur-concavity of $\Psi(\alpha, \boldsymbol{\beta}, \lambda, x, \phi_1)$ follows from Lemma 2.3. According to Lemma 2.4, $(\beta_1, \ldots, \beta_n) \overset{\mathrm{m}}{\succeq} (\beta_1^*, \ldots, \beta_n^*)$ implies $\Psi(\alpha, \boldsymbol{\beta}, \lambda, x, \phi_1) \leq \Psi(\alpha, \boldsymbol{\beta^*}, \lambda, x, \phi_1)$. On the other hand, since $\psi_2 \circ \phi_1$ is super-additive by Lemma 2.5, we have $\Psi(\alpha, \boldsymbol{\beta^*}, \lambda, x, \phi_1) \leq \Psi(\alpha, \boldsymbol{\beta^*}, \lambda, x, \phi_2)$. So, it holds that

$$\Psi(\alpha, \boldsymbol{\beta}, \lambda, x, \phi_1) \leq \Psi(\alpha, \boldsymbol{\beta^*}, \lambda, x, \phi_1) \leq \Psi(\alpha, \boldsymbol{\beta^*}, \lambda, x, \phi_2).$$

That is, $X_{n:n} \geq_{\mathrm{st}} X_{n:n}^*$. $\square$

Remark 3.9. *It is to be noted that the super-additive assumption of $\psi_2 \circ \phi_1$ is satisfied by many members of Archimedean copulas. For example, the Archimedean survival copula with generators (i) $\phi_1(t) = e^{1-(1+t)^{\frac{1}{\theta}}}$ and $\phi_2(t) = \dfrac{\theta}{\log(e^\theta + t)}$, where $0 < \theta \leq 1$, (ii) $\phi_1(t) = \dfrac{\theta}{\log(e^\theta + t)}$ and $\phi_2(t) = \log(e+t)^{\frac{-1}{\theta}}$, where $\theta > 1$, and (iii) $\phi_1(t) = e^{1-(1+t)^{\frac{1}{\theta_1}}}$ and $\phi_2(t) = e^{1-(1+t)^{\frac{1}{\theta_2}}}$, where $\theta_2 \geq \theta_1 \geq 1$, satisfy super-additivity.*

It is well recognized that the maximum order statistics correspond to the lifetimes of parallel systems. Based on Lemma 2.5, the assertion that "$\psi_2 \circ \phi_1$ is super-additive" implies that a copula with generator ϕ_2 exhibits stronger positive dependence than one with generator ϕ_1. Therefore, invoking Theorem 3.8, one can conclude that, under an Archimedean copula capturing the dependence structure among positively dependent components following NEW distributions, a shape parameter vector $\boldsymbol{\beta}$ that is more heterogeneous in the sense of majorization results in a more reliable parallel system. Specifically, the system lifetime in this scenario is stochastically larger, highlighting the favorable impact of the reduced heterogeneity in shape parameters on the reliability of parallel systems.

We now provide an example illustrating that under the conditions of Theorem 3.8, for two parallel systems whose components are mutually dependent via a Gumbel–Hougaard copula with specified parameters and following NEWs, the survival function of one system is smaller than that of the other.

Example 3.10. *Consider two three-component parallel systems A and B. Let the lifetimes of the components of the first system be denoted by* (X_1, X_2, X_3) *and that of the second system be denoted by* (X_1^*, X_2^*, X_3^*). *Assume that the lifetimes follow NEW models. Let* $(\beta_1, \beta_2, \beta_3) = (1, 2, 6)$, $(\beta_1^*, \beta_2^*, \beta_3^*) = (2, 3, 4)$, $\alpha = 2$, $\lambda = 3$, *and* $\phi_1(x) = e^{-x^{1/\theta_1}}$ *is for the Gumbel copula with parameter* $\theta_1 = 2$, $\phi_2(x) = e^{-x^{1/\theta_2}}$ *is for the Gumbel copula with parameter* $\theta_2 = 4$. *All the conditions of Theorem 3.8 are satisfied. Thus, as an application of Theorem 3.8, we conclude that system A has better performance than system B in the sense of the usual stochastic ordering, as can be seen in Figure 3.*

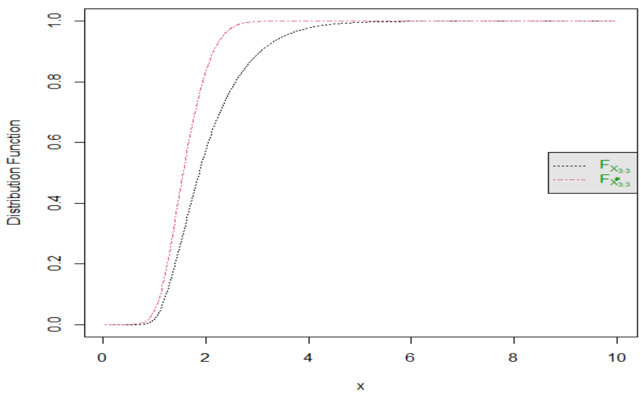

Figure 3: Plot of the distribution functions of $X_{3:3}$ and $X_{3:3}^*$.

Nadeb et al. (2021) obtained the following theorem for the comparison of parallel systems under the usual stochastic order.

Theorem 3.11. *Let* $\boldsymbol{X}$ *and* $\boldsymbol{X}^*$ *be two vectors of continuous random variables with* $X_i \sim F(x; \lambda_i)$ *and* $X_i^* \sim F(x; \lambda_i^*)$, $i = 1, \ldots, n$. *Assume that their associated copulas are Archimedean with the respective generators* ϕ_1 *and* ϕ_2. *If* ϕ_1 *or* ϕ_2 *is log-convex,* $\psi_2 \circ \phi_1$ *is super-additive, and for any* x *the function* $F(x; \lambda)$ *is decreasing (increasing) and log-concave in* $\boldsymbol{\lambda}$, *then* $(\lambda_1, \ldots, \lambda_n) \succeq_{\mathrm{w}} [\overset{\mathrm{w}}{\succeq}](\lambda_1^*, \ldots, \lambda_n^*)$ *implies* $X_{n:n} \geq_{\mathrm{st}} X_{n:n}^*$.

The following corollary provides a comparison between the largest order statistics arising from two sets of random variables, when the marginal distributions belong to the NEW distribution.

Corollary 3.12. *Suppose, for* $\boldsymbol{X} \sim \mathrm{NEW}(\boldsymbol{\alpha}, \beta, \lambda, \phi_1)$ *and* $\boldsymbol{X}^* \sim \mathrm{NEW}(\boldsymbol{\alpha}^*, \beta, \lambda, \phi_2)$, ϕ_1 *or* ϕ_2 *is log-convex and* $\psi_2 \circ \phi_1$ *is super-additive, then* $(\alpha_1, \ldots, \alpha_n) \overset{\mathrm{w}}{\succeq} (\alpha_1^*, \ldots, \alpha_n^*)$ *implies* $X_{n:n} \geq_{\mathrm{st}} X_{n:n}^*$.

Proof. We can see that $F(x, \alpha)$ is increasing and log-concave in α. Hence, Theorem 3.11 immediately implies the result. □

4 A Monte Carlo Simulation Study

To further illustrate the applicability of Theorem 3.1, a Monte Carlo simulation study is conducted for the smallest order statistic under three different Archimedean dependence structures, namely the Clayton, Gumbel, and Frank copulas.

Let $\alpha = 1$, $\lambda = 1$, and consider the two vectors $\boldsymbol{\beta} = (4, 3, 2, 1, 0.8)$ and $\boldsymbol{\beta}^* = (3.5, 2.8, 2, 1.5, 1)$. Both vectors are arranged in decreasing order and satisfy

$$\sum_{i=1}^{5} \beta_i = 4 + 3 + 2 + 1 + 0.8 = 10.8,$$
$$\sum_{i=1}^{5} \beta_i^* = 3.5 + 2.8 + 2 + 1.5 + 1 = 10.8.$$

Furthermore,

$$\begin{gathered} 4 \geq 3.5, \\ 4 + 3 \geq 3.5 + 2.8, \\ 4 + 3 + 2 \geq 3.5 + 2.8 + 2, \\ 4 + 3 + 2 + 1 \geq 3.5 + 2.8 + 2 + 1.5. \end{gathered}$$

Consequently, $\boldsymbol{\beta} \succeq^{\mathrm{m}} \boldsymbol{\beta}^*$. For $i = 1, \ldots, 5$, the marginal survival functions of X_i and X_i^*, respectively, are given by

$$\begin{aligned} \overline{F}_i(x) &= \exp\left\{-\alpha x^{\beta_i} \exp\left(-\frac{\lambda}{x}\right)\right\}, \qquad x > 0, \\ \overline{F}_i^*(x) &= \exp\left\{-\alpha x^{\beta_i^*} \exp\left(-\frac{\lambda}{x}\right)\right\}, \qquad x > 0. \end{aligned}$$

For an Archimedean survival copula with generator ϕ and inverse generator $\psi = \phi^{-1}$, the survival function of the smallest order statistic is

$$\overline{F}_{X_{1:n}}(x) = \phi\left(\sum_{i=1}^{n} \psi\left(\overline{F}_i(x)\right)\right).$$

Therefore, for the first random vector,

$$\overline{F}_{X_{1:5}}(x) = \phi\left[\sum_{i=1}^{5} \psi\left\{\exp\left(-\alpha x^{\beta_i} \exp\left(-\frac{\lambda}{x}\right)\right)\right\}\right],$$

whereas for the second random vector,

$$\overline{F}_{X_{1:5}^*}(x) = \phi\left[\sum_{i=1}^{5} \psi\left\{\exp\left(-\alpha x^{\beta_i^*} \exp\left(-\frac{\lambda}{x}\right)\right)\right\}\right].$$

We consider the following three Archimedean survival copulas:

(i) The Clayton family with the generator and its inverse as

$$\phi_{\mathrm{C}}(t) = (1 + t)^{-1/\theta}, \qquad \psi_{\mathrm{C}}(u) = u^{-\theta} - 1, \qquad \theta > 0.$$

Here, we take $\theta = 2$.
(ii) The Gumbel family with the generator and its inverse as

$$\phi_{\mathrm{G}}(t) = \exp\left(-t^{1/\theta}\right), \qquad \psi_{\mathrm{G}}(u) = (-\log u)^{\theta}, \qquad \theta \geq 1.$$

Here, we take $\theta = 2$.
(iii) The Frank family with the generator and its inverse as

$$\begin{aligned}\phi_{\mathrm{F}}(t) &= -\frac{1}{\theta}\log\left[1-(1-e^{-\theta})e^{-t}\right], \qquad \theta > 0,\\ \psi_{\mathrm{F}}(u) &= -\log\left[\frac{1-e^{-\theta u}}{1-e^{-\theta}}\right].\end{aligned}$$

Here, we take $\theta = 5$.

For each copula family, the same generator and the same dependence parameter are used for both random vectors. Thus,

$$\phi_1 = \phi_2 = \phi, \qquad \psi_2 \circ \phi_1 = \psi \circ \phi = I,$$

where I denotes the identity function. Hence, the super-additivity condition required in Theorem 3.1 is satisfied with equality. For each copula family, $B = 10{,}000$ Monte Carlo replications are performed. In each replication, a random vector $(U_1, \ldots, U_5)$ is generated from the corresponding Archimedean survival copula. The components of the first random vector are then obtained from $U_i = \overline{F}_i(X_i)$, $i = 1, \ldots, 5$, or, equivalently, by solving

$$\alpha X_i^{\beta_i}\exp(-\frac{\lambda}{X_i}) = -\log U_i.$$

Similarly, the components of the second random vector are obtained from

$$U_i^* = \overline{F}_i^*(X_i^*), \qquad i = 1, \ldots, 5.$$

For each replication, the smallest order statistics

$$X_{1:5} = \min\{X_1, \ldots, X_5\}, \qquad X_{1:5}^* = \min\{X_1^*, \ldots, X_5^*\}$$

are calculated. The empirical survival functions are estimated by

$$\begin{aligned}\widehat{\overline{F}}_{X_{1:5}}(x) &= \frac{1}{B}\sum_{b=1}^{B} I\left(X_{1:5}^{(b)} > x\right),\\ \widehat{\overline{F}}_{X_{1:5}^*}(x) &= \frac{1}{B}\sum_{b=1}^{B} I\left(X_{1:5}^{*(b)} > x\right).\end{aligned}$$

The theoretical and empirical survival probabilities at selected values of x are reported in Table 1. Here, $\widehat{\Delta}(x) = \widehat{\overline{F}}_{X_{1:5}^*}(x) - \widehat{\overline{F}}_{X_{1:5}}(x)$.

The theoretical differences are nonnegative at all reported values of x, as predicted by Theorem 3.1. The empirical results are also consistent with this ordering. The small discrepancies between the theoretical and empirical values are attributable to Monte

Table 1: Theoretical and empirical survival probabilities for the three copula families.

Copula	x	$\overline{F}_{X_{1:5}}(x)$	$\overline{F}_{X^*_{1:5}}(x)$	$\widehat{\overline{F}}_{X_{1:5}}(x)$	$\widehat{\overline{F}}_{X^*_{1:5}}(x)$	$\widehat{\Delta}(x)$
Clayton	0.50	0.8349	0.8515	0.8401	0.8554	0.0153
	0.75	0.5997	0.6067	0.6018	0.6088	0.0070
	1.00	0.3942	0.3942	0.3951	0.3951	0.0000
	1.25	0.2191	0.2286	0.2151	0.2250	0.0099
	1.50	0.0662	0.0947	0.0693	0.0955	0.0262
	2.00	0.0001	0.0010	0.0000	0.0011	0.0011
Gumbel	0.50	0.8957	0.9118	0.9005	0.9152	0.0147
	0.75	0.7029	0.7139	0.7061	0.7168	0.0107
	1.00	0.4393	0.4393	0.4484	0.4484	0.0000
	1.25	0.1734	0.1838	0.1710	0.1824	0.0114
	1.50	0.0305	0.0432	0.0304	0.0432	0.0128
	2.00	1.3e-5	1.71e-4	0.0001	0.0003	0.0002
Frank	0.50	0.8470	0.8623	0.8511	0.8664	0.0153
	0.75	0.6374	0.6450	0.6436	0.6518	0.0082
	1.00	0.4174	0.4174	0.4200	0.4200	0.0000
	1.25	0.1812	0.1925	0.1837	0.1939	0.0102
	1.50	0.0261	0.0398	0.0268	0.0391	0.0123
	2.00	5e-7	1.24e-5	0.0000	0.0000	0.0000

Carlo variation. For example, under the Clayton copula at $x = 1.5$, $\overline{F}_{X_{1:5}}(1.5) = 0.0662$, whereas $\overline{F}_{X^*_{1:5}}(1.5) = 0.0947$. Similarly, the corresponding empirical estimates are $\widehat{\overline{F}}_{X_{1:5}}(1.5) = 0.0693$ and $\widehat{\overline{F}}_{X^*_{1:5}}(1.5) = 0.0955$. Thus, the simulation agrees with $X_{1:5} \leq_{\text{st}} X^*_{1:5}$.

The same behavior is observed for the Gumbel and Frank copulas. For instance, at $x = 1.5$, the theoretical survival probabilities under the Gumbel copula are 0.0305 and 0.0432, respectively, while under the Frank copula they are 0.0261 and 0.0398, respectively. Hence, in all three dependence structures, the survival probability of $X^*_{1:5}$ is greater than that of $X_{1:5}$. Overall, the numerical results are consistent with the stochastic ordering established in Theorem 3.1 and demonstrate that the ordering is preserved under three different Archimedean dependence structures.

An analogous simulation study can be performed for the largest order statistic. In accordance with the corresponding theorem, the same majorization condition yields

$$X_{n:n} \geq_{\text{st}} X^*_{n:n},$$

and the corresponding numerical results exhibit the same consistency with the theoretically established stochastic ordering.

5 Concluding Remarks

This is the first attempt to study some stochastic comparisons of order statistics from dependent and heterogeneous samples having the NEW family. We derived the usual stochastic order for the smallest order statistic of samples having the new extended Weibull family and Archimedean survival copulas. Similar ordering results are also obtained for the largest order statistic of samples having the new extended Weibull family and Archimedean copulas. Several examples are provided to illustrate the established results. A useful practical implication of our results concerns parallel systems. If the system designer has control over the shape parameters β_i, then increasing the heterogeneity among these parameters can lead to a stochastically larger system lifetime. Thus, in the design of parallel systems, deliberately introducing an appropriate degree of heterogeneity in the component shape parameters may improve system reliability and prolong the overall system lifetime. This provides a potentially useful design insight for applications in which the component characteristics can be controlled or engineered. We conclude by noting some limitations and open problems. Our results rely on specific conditions, including the log-convexity of the copula generator, or equivalently on the required log-convexity properties of ϕ_1 or ϕ_2. This condition is satisfied by some important Archimedean copulas, such as the Gumbel and Clayton copulas, but not by all Archimedean copulas, such as the Frank copula. Moreover, verifying the required super-additivity condition may be difficult for general copula families. Our framework also assumes a common copula for all components, which may not adequately describe systems with heterogeneous dependence structures. Relaxing these assumptions and studying the corresponding results under other stochastic orders, such as the hazard rate, reversed hazard rate, and likelihood ratio orders, are interesting directions for future research.

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