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Research Paper

Generalization of reliability in the odd Lindley-G family

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Abstract: This study presents a generalized numerical framework for evaluating the reliability criterion in the odd Lindley-G family of distributions. Although the odd Lindley-G family provides a flexible mechanism for extending classical baseline models, the existing analytical formulation for reliability estimation suffers from two critical limitations: it is restricted to cases where the strength and stress variables share an identical baseline distribution, and its series expansion diverges even under homogeneous baseline conditions. To address these limitations, we develop a robust numerical integration framework that accurately computes reliability for arbitrary and distinct baseline distributions within the odd Lindley-G family. The proposed method is validated through an extensive simulation study and a practical engineering application involving strength and stress data with different baseline distributions. Our findings confirm that the numerical framework not only generalizes the applicability of the odd Lindley-G family but also delivers a stable, accurate, and computationally efficient tool for advanced reliability analysis.

Keywords: Numerical integration; Odd Lindley-G family; Stress-strength reliability.
Mathematics Subject Classification (2010): 62N05.

1 Introduction

Flexible statistical distributions have gained considerable attention in science and engineering, as classical models often fail to capture complex features such as heavy tails and non-uniform hazard rates. This has motivated the development of generalized

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families that extend standard models via additional shape or scale parameters, thereby enhancing modeling flexibility and goodness of fit in fields such as reliability, survival analysis, economics, and medical research.

Among the most influential G-families–which are constructed by adding one or more parameters to a baseline distribution $G(x)$–are the Marshall-Olkin generated family (MO-G) by Marshall and Olkin (1997), the beta-G by Eugene et al. (2002), the gamma-G by Zografos and Balakrishnan (2009), and the Kumaraswamy-G (Kw-G) by Cordeiro and de Castro (2011). Other notable extensions include the Weibull-G by Bourguignon et al. (2014), the exponentiated half-logistic family by Cordeiro et al. (2014), and more recently the odd log-logistic Lindley-G family (Alizadeh et al., 2020) and the Marshall-Olkin odd Burr III-G family (Afify et al., 2021). While some of these families have been investigated for stress-strength reliability, as discussed below, others remain unexplored in this context.

For stress-strength reliability, the quantity $R = P(X_1 > X_2)$ is fundamental where X_1 represents strength and X_2 represents stress. Among the aforementioned G-families, the MO-G family has been extensively studied for this purpose (Kotz et al., 2003; Cüran and Kızılaslan, 2025; Gómez-Déniz et al., 2024). The beta-G family has also received attention, with Jafari et al. (2014) deriving explicit expressions for R, and subsequent work extending these results to multicomponent systems (Chaturvedi et al., 2022). The gamma-G family has been studied for stress-strength reliability by Pal and Tiensuwan (2015), who derived various properties including moment generating functions and Rényi entropy. The Kw-G family has been investigated for reliability in multicomponent stress-strength models (Wang et al., 2020; Saini and Garg, 2022). However, for the odd Lindley-G (OL-G) family introduced by Gomes-Silva et al. (2017), the reliability formulation remains problematic and has not been systematically studied.

The Lindley distribution (Lindley, 1958) is well known for its applications in reliability and longevity data analysis. Ghitany et al. (2008) showed that the Lindley distribution exhibits a non-constant hazard rate, making it more flexible than the exponential distribution. Building on the T-X framework (Alzaatreh et al., 2013), Gomes-Silva et al. (2017) introduced the OL-G family using the odds function $\frac{G(x)}{1-G(x)}$ as the generator and the Lindley distribution as the generating distribution. The resulting family adds a parameter $a > 0$ to each baseline distribution, creating a flexible class that can model various hazard shapes while maintaining mathematical tractability. The authors introduced several submodels including the odd Lindley-Weibull (OLW), the odd Lindley-Kumaraswamy (OLKw), the odd Lindley-half-logistic (OLHL), and the odd Lindley-Burr XII (OLBXII), and discussed parameter estimation via maximum likelihood (ML).

The OL-G family offers several advantages over existing G-families for reliability analysis. Unlike the beta-G family, which is largely restricted to unimodal hazard shapes (Sibuya, 1998; Cordeiro and de Castro, 2011), and unlike the gamma-G family, which often requires numerical integration for its cumulative distribution function (CDF), the OL-G family accommodates a wide range of hazard shapes—increasing, decreasing, bathtub, and upside-down bathtub—while maintaining a closed-form CDF (Gomes-Silva et al., 2017). This flexibility is especially valuable in reliability engineering, where failure rates often exhibit complex patterns.

It is important to note that the divergence issue we address in this paper is not common to all G-families. While the series expansion for R converges in the MO-G family under mild conditions (Kotz et al., 2003; Cüran and Kızılaslan, 2025) and in the beta-G family for all parameter values (Jafari et al., 2014; Chaturvedi et al., 2022), the OL-G series diverges because its terms fail to vanish along an admissible sequence, as shown in Section 3. This limitation, together with the restrictive assumption that stress and strength share an identical baseline distribution, severely restricts the practical applicability of the OL-G family. In real-world problems, stress and strength arise from different physical processes and often follow different distributions.

This paper addresses this gap by developing a general numerical framework for computing stress-strength reliability in the OL-G family without the restrictive assumption of identical baseline distributions. Our contributions are threefold: (i) we provide a rigorous proof that the series expansion in Gomes-Silva et al. (2017) diverges for all $a_1, a_2 > 0$; (ii) we propose a robust numerical integration method based on $R = \int f_1(x)F_2(x)\,dx$, which accommodates arbitrary and distinct baseline distributions; and (iii) we validate the framework through extensive simulations and a practical engineering example involving OL-Weibull and OL-lognormal distributions.

The remainder of this paper is organized as follows. Section 2 provides a brief review of the OL-G family. Section 3 demonstrates the divergence of the existing series expansion. Section 4 presents the proposed numerical integration framework. Section 5 reports the simulation study and engineering application. Section 6 concludes by discussing the conclusions, limitations, and future directions.

2 A brief introduction to the OL-G family

The OL-G family of distributions is a flexible generator of continuous distributions constructed using the T-X family framework. Let $G(x;\boldsymbol{\xi})$ be a baseline CDF with corresponding probability density function (PDF) $g(x;\boldsymbol{\xi})$, where $\boldsymbol{\xi}$ denotes a vector of parameters. The CDF of the OL-G family is defined by

$$F(x;a,\boldsymbol{\xi}) = 1 - \frac{a+\overline{G}(x;\boldsymbol{\xi})}{(1+a)\,\overline{G}(x;\boldsymbol{\xi})}\,\exp\left(-a\,\frac{G(x;\boldsymbol{\xi})}{\overline{G}(x;\boldsymbol{\xi})}\right), \quad x \in \mathbb{R}, \tag{1}$$

where $a > 0$ is an additional shape parameter and $\overline{G}(x;\boldsymbol{\xi}) = 1 - G(x;\boldsymbol{\xi})$ is the baseline survival function. The corresponding PDF is given by

$$f(x;a,\boldsymbol{\xi}) = \frac{a^2}{1+a}\,\frac{g(x;\boldsymbol{\xi})}{\left[\overline{G}(x;\boldsymbol{\xi})\right]^3}\,\exp\left(-a\,\frac{G(x;\boldsymbol{\xi})}{\overline{G}(x;\boldsymbol{\xi})}\right). \tag{2}$$

For a random variable X following an OL-G distribution, we write $X \sim$ OL-G$(a,\boldsymbol{\xi})$. To aid interpretation, the CDF in (1) can be viewed as a baseline survival function $\overline{G}(x)$ modified by an odds ratio $G(x)/\overline{G}(x)$ and a Lindley-type adjustment term $\exp\{-a\,G(x)/\overline{G}(x)\}$. This structure enables the OL-G family to accommodate a wide range of hazard shapes, including increasing, decreasing, bathtub, and upside-down bathtub forms.

The proposed framework relies on several assumptions. The strength variable X_1 and the stress variable X_2 are assumed independent. Both baseline distributions G_1

and G_2 are continuous, with strictly positive shape parameters ($a > 0$) and intersecting supports. Additionally, the fitted OL-G distributions must adequately capture the empirical characteristics of the data, as confirmed by goodness-of-fit tests. Several well-known distributions emerge as special sub-models when specific baseline distributions G are chosen. Notable examples include

- **OLW by Yousof et al. (2018):** $G(x) = 1 - \exp\big(-(\lambda x)^\alpha\big)$, $\alpha, \lambda > 0$.
- **OLKw by Samson et al. (2020):** $G(x) = 1 - (1 - x^\alpha)^\beta$, $0 < x < 1$, $\alpha, \beta > 0$.
- **OLHL by Cordeiro et al. (2014):** $G(x) = \frac{1-e^{-x}}{1+e^{-x}}$, $x > 0$.
- **OLBXII by Yousof et al. (2017):** $G(x) = 1 - \big(1 + (x/s)^c\big)^{-k}$, $x > 0$, $s, c, k > 0$.

3 On the divergence of the series representation

Consider the reliability measure $R = P(X_1 > X_2)$, where X_1 and X_2 are independent random variables from the OL-G family with the same baseline distribution G and positive parameters a_1 and a_2. Gomes-Silva et al. (2017) proposed a series representation as

$$R = \sum_{i,j,k,l=0}^{\infty} \frac{p_{i,j}(a_1) q_{k,l}(a_2)}{i+j+k+l+2}, \tag{3}$$

where

$$p_{i,j}(a_1) = \frac{(-1)^j a_1^{j+2} \Gamma(i+j+3)}{i!\, j!\, (a_1+1)\Gamma(j+3)}, \tag{4}$$

$$q_{k,l}(a_2) = \frac{(-1)^l a_2^{l+2} \Gamma(k+l+3)}{k!\, l!\, (a_2+1)(k+l+1)\Gamma(l+3)}. \tag{5}$$

We show that the fourfold series in (3) does not converge in the sense of Pringsheim for any $a_1, a_2 > 0$.

3.1 Pringsheim convergence

Let $A_{i,j,k,l} = \frac{p_{i,j}(a_1) q_{k,l}(a_2)}{i+j+k+l+2}$, and define the rectangular partial sums by

$$S_{n_1,n_2,n_3,n_4} = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \sum_{k=0}^{n_3} \sum_{l=0}^{n_4} A_{i,j,k,l}.$$

The series is said to converge in the sense of Pringsheim if there exists a finite number S such that

$$S_{n_1,n_2,n_3,n_4} \longrightarrow S, \qquad \text{whenever} \qquad \min\{n_1,n_2,n_3,n_4\} \longrightarrow \infty. \tag{6}$$

The following necessary condition will be used.

Lemma 3.1. *If the series in* (3) *converges in the sense of Pringsheim, then*

$$A_{i,j,k,l} \longrightarrow 0, \qquad \text{as} \qquad \min\{i,j,k,l\} \longrightarrow \infty. \tag{7}$$

Proof. For $i, j, k, l \geq 1$, the term $A_{i,j,k,l}$ is the fourfold rectangular increment of the partial-sum field as $A_{i,j,k,l} = \Delta_i \Delta_j \Delta_k \Delta_l S_{i,j,k,l}$. If $\min\{i, j, k, l\} \to \infty$, then every partial sum occurring in this fourfold increment has all four indices tending to infinity. Under (6), each of these partial sums therefore converges to S. Their alternating sum consequently tends to zero, proving (7). □

Theorem 3.2. *For every $a_1, a_2 > 0$, the series in* (3) *does not converge in the sense of Pringsheim.*

Proof. Let $m_N = \lfloor \log N \rfloor$, $N \geq 3$, and consider the admissible sequence of multi-indices $(i, j, k, l) = (N, m_N, N, m_N)$. Since $m_N \to \infty$, we have

$$\min\{N, m_N, N, m_N\} = m_N \to \infty.$$

For this choice, $j = l = m_N$, so the alternating signs in (4) and (5) cancel. Hence $A_{N,m_N,N,m_N} > 0$, and direct substitution gives

$$A_{N,m_N,N,m_N} = \frac{a_1^{m_N+2} a_2^{m_N+2}}{2(a_1+1)(a_2+1)} \frac{\Gamma(N+m_N+3)^2}{(N!)^2 (m_N!)^2 \Gamma(m_N+3)^2 (N+m_N+1)^2}.$$

Since $\frac{\Gamma(N+m_N+3)}{\Gamma(N+1)} = \prod_{r=1}^{m_N+2}(N+r) \geq N^{m_N+2}$ and $\Gamma(m_N+3) = (m_N+2)!$, it follows that

$$A_{N,m_N,N,m_N} \geq C(a_1, a_2) \frac{(a_1 a_2)^{m_N} N^{2m_N+4}}{(m_N!)^2 ((m_N+2)!)^2 (N+m_N+1)^2}, \tag{8}$$

where $C(a_1, a_2) = \frac{a_1^2 a_2^2}{2(a_1+1)(a_2+1)} > 0$. For sufficiently large N, $m_N \leq N$, and therefore $N + m_N + 1 \leq 3N$. Using $m_N! \leq m_N^{m_N}$ and $(m_N+2)! \leq (m_N+2)^{m_N+2}$, we obtain

$$A_{N,m_N,N,m_N} \geq \frac{C(a_1, a_2)}{9} \frac{(a_1 a_2)^{m_N} N^{2m_N+2}}{m_N^{2m_N} (m_N+2)^{2m_N+4}}. \tag{9}$$

Set $L_N = \log N$. Taking logarithms in (9) yields

$$\begin{aligned} \log A_{N,m_N,N,m_N} \quad \geq \quad & C_0 + m_N \log(a_1 a_2) + (2m_N+2) L_N \\ & -2m_N \log m_N - 2(m_N+2)\log(m_N+2), \end{aligned}$$

where C_0 is independent of N. Because

$$m_N = L_N + O(1), \qquad \log m_N = \log L_N + o(1), \qquad \log(m_N+2) = \log L_N + o(1),$$

we have

$$\log A_{N,m_N,N,m_N} \geq 2L_N^2 - 4L_N \log L_N + O(L_N). \tag{10}$$

Here, $m_N \log(a_1 a_2) = O(L_N)$, for every fixed $a_1, a_2 > 0$. Since $\frac{L_N \log L_N}{L_N^2} \longrightarrow 0$, the right-hand side of (10) tends to $+\infty$. Thus, $A_{N,m_N,N,m_N} \longrightarrow +\infty$. Therefore, along a sequence for which $\min\{i, j, k, l\} \longrightarrow \infty$, the terms of the series do not tend to zero. This contradicts the necessary condition in Lemma 3.1. Hence the series in (3) does not converge in the sense of Pringsheim for any $a_1, a_2 > 0$. Moreover,

$$|A_{N,m_N,N,m_N}| = A_{N,m_N,N,m_N} \longrightarrow +\infty,$$

so the terms of the series of absolute values also fail to tend to zero. Consequently, the series is not absolutely convergent. □

Remark 3.3. *The divergence established in Theorem 3.2 concerns only the fourfold series representation, not the reliability measure R itself. The latter remains a valid probability and can be computed directly from its defining integral, which always yields a value in the interval $[0,1]$. Moreover, the series in question is not a standard power series; it is a multiple numerical series whose terms depend on the parameters a_1 and a_2. Therefore, the classical concept of a radius of convergence does not apply here. Finally, as shown in the proof, the series is not absolutely convergent, since the terms of the absolute series fail to tend to zero along the admissible sequence. Thus, the original fourfold series cannot be assigned a value through Pringsheim convergence or absolute convergence.*

3.2 Numerical illustration

The theoretical result above is independent of numerical truncation. Nevertheless, finite rectangular truncations provide useful illustrations of the severe instability caused by the divergent representation. For $a_1 = a_2 = 1$, the rectangular truncations

$$S_N = \sum_{i,j,k,l=0}^{N} \frac{p_{i,j}(1)q_{k,l}(1)}{i+j+k+l+2},$$

exhibit substantial oscillation rather than stabilization toward a finite value:

N	5	8	10	12	20	80
S_N	0.7753	0.8114	0.5701	0.3663	0.2441	1.1665

These values merely illustrate the numerical behavior of finite truncations and are not used as a proof of divergence. A further illustration is obtained from the complementarity property of reliability probabilities. For independent random variables, $R(a_1, a_2) + R(a_2, a_1) = 1$. Finite truncations of the divergent representation need not preserve this identity. For example, a particular finite truncation may produce $R(1.5, 0.8) + R(0.8, 1.5) = 1.2076 \neq 1$, and may also produce values outside the admissible range $[0, 1]$, such as $R(2.0, 1.0) = 5.6048 > 1$.

3.3 Growth of selected coefficients

For $j = l = 0$, the coefficients simplify to

$$\begin{aligned} p_{i,0}(a_1) &= \frac{a_1^2}{2(a_1+1)}(i+1)(i+2), \\ q_{k,0}(a_2) &= \frac{a_2^2}{2(a_2+1)}(k+2). \end{aligned}$$

Thus, $p_{i,0}(a_1) = O(i^2)$ and $q_{k,0}(a_2) = O(k)$. Along the diagonal $i = k$,

$$\frac{p_{i,0}(a_1)q_{i,0}(a_2)}{2i+2} = \frac{a_1^2 a_2^2}{8(a_1+1)(a_2+1)}(i+2)^2,$$

which grows quadratically in i. This is consistent with the asymptotic behavior $T_{i,k} \sim Ci^2k/(i+k)$, where $T_{i,k}$ denotes the corresponding term of the series.

Table 1 provides representative values for $a_1 = a_2 = 1$. The table is included solely to visualize the growth of selected coefficients and the corresponding diagonal terms; the analytical proof of divergence is given in Theorem 1.

Table 1: Growth of selected coefficients for $a_1 = a_2 = 1$.

i	$p_{i,0}$	$q_{i,0}$	$\frac{p_{i,0}q_{i,0}}{2i+2}$
1	1.500	0.750	0.281
2	3.000	1.000	0.500
5	10.500	1.750	1.531
10	33.000	3.000	4.500
20	115.500	5.500	15.125
50	663.000	13.000	84.500

4 Numerical integration framework for generalized reliability estimation

Having demonstrated the divergence of the series representation in (3), we now propose a direct numerical integration framework for computing the stress-strength reliability R within the OL-G family. This approach avoids the divergent series expansion and allows the stress and strength variables to have different baseline distributions.

4.1 General formulation

Let X_1 and X_2 be two independent continuous random variables such that $X_1 \sim$ OL-G$(a_1, \boldsymbol{\xi}_1)$, and $X_2 \sim$ OL-G$(a_2, \boldsymbol{\xi}_2)$, with corresponding baseline distributions $G_1(x; \boldsymbol{\xi}_1)$ and $G_2(x; \boldsymbol{\xi}_2)$. The baseline distributions and their parameter vectors are not required to be identical.

The stress-strength reliability is defined as

$$R = P(X_1 > X_2) = \int_{\mathcal{D}_1} f_1(x; a_1, \boldsymbol{\xi}_1) F_2(x; a_2, \boldsymbol{\xi}_2)\, dx,$$

where $\mathcal{D}_1$ denotes the support of X_1, $f_1(\cdot; a_1, \boldsymbol{\xi}_1)$ is the PDF of X_1, and $F_2(\cdot; a_2, \boldsymbol{\xi}_2)$ is the CDF of X_2.

Using the OL-G PDF and CDF given in (2) and (1), respectively, we obtain the computable integral

$$\begin{aligned} R(a_1, a_2, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2) &= \frac{a_1^2}{1+a_1} \int_{\mathcal{D}_1} \frac{g_1(x; \boldsymbol{\xi}_1)}{\left[\overline{G}_1(x; \boldsymbol{\xi}_1)\right]^3} \exp\left\{-a_1 \frac{G_1(x; \boldsymbol{\xi}_1)}{\overline{G}_1(x; \boldsymbol{\xi}_1)}\right\} \\ &\quad \times \left[1 - \frac{a_2 + \overline{G}_2(x; \boldsymbol{\xi}_2)}{(1+a_2)\overline{G}_2(x; \boldsymbol{\xi}_2)} \exp\left\{-a_2 \frac{G_2(x; \boldsymbol{\xi}_2)}{\overline{G}_2(x; \boldsymbol{\xi}_2)}\right\}\right] dx, \end{aligned}$$

where $\overline{G}_r(x; \boldsymbol{\xi}_r) = 1 - G_r(x; \boldsymbol{\xi}_r)$, $r = 1, 2$.

4.2 Numerical evaluation

The integral in (4.1) is evaluated directly using the adaptive numerical integration procedure implemented in the R function `stats::integrate()`, which is based on the QUADPACK routines (Piessens et al., 1983). This procedure adaptively subdivides the integration interval and provides an estimate of the numerical integration error.

For unbounded supports, we use a quantile-based truncation criterion. Let $\epsilon_{\text{tr}} > 0$ and define

$$L = F_1^{-1}(\epsilon_{\text{tr}}), \qquad U = F_1^{-1}(1 - \epsilon_{\text{tr}}), \tag{11}$$

where F_1 is the CDF of X_1. The corresponding truncated reliability is

$$R_{[L,U]} = \int_L^U f_1(x; a_1, \boldsymbol{\xi}_1) F_2(x; a_2, \boldsymbol{\xi}_2)\, dx.$$

Since $0 \leq F_2(x; a_2, \boldsymbol{\xi}_2) \leq 1$, the truncation error satisfies

$$\left|R - R_{[L,U]}\right| \leq \int_{-\infty}^{L} f_1(x)\, dx + \int_U^{\infty} f_1(x)\, dx = P(X_1 < L) + P(X_1 > U) = 2\epsilon_{\text{tr}}. \tag{12}$$

In the numerical calculations, we set $\epsilon_{\text{tr}} = 10^{-8}$ for the tail truncation and use a relative quadrature tolerance of $\epsilon_{\text{quad}} = 10^{-8}$. These two tolerances control different sources of numerical approximation: ϵ_{tr} controls the probability mass omitted by truncating an unbounded support, whereas ϵ_{quad} controls the error from the numerical quadrature. For bounded supports, the integral is evaluated directly over the support of X_1, without tail truncation.

4.3 Advantages over the series representation

The proposed numerical framework avoids the divergent fourfold series representation established in Section 3 by evaluating the defining reliability integral directly. Consequently, the divergence of the series does not affect the definition of R or its direct numerical evaluation.

In addition, the formulation does not require the baseline distributions of X_1 and X_2 to be identical. Thus,

$$G_1(x; \boldsymbol{\xi}_1) \neq G_2(x; \boldsymbol{\xi}_2),$$

is permitted, allowing reliability to be evaluated for heterogeneous OL-G models, such as OL-Weibull versus OL-Gamma or OL-lognormal versus OL-Fréchet, provided that the corresponding distributions are well-defined and the integral can be evaluated numerically.

4.4 Implementation outline

The computation of R via (4.1) consists of the following steps:

i. **Baseline Specification:** Specify the baseline CDFs $G_1(\cdot; \boldsymbol{\xi}_1)$ and $G_2(\cdot; \boldsymbol{\xi}_2)$, together with their corresponding PDFs $g_1(\cdot; \boldsymbol{\xi}_1)$ and $g_2(\cdot; \boldsymbol{\xi}_2)$.

ii. **Domain Determination:** Determine the support $\mathcal{D}_1$ of X_1. For an unbounded support, determine the finite limits L and U according to (11), using $\epsilon_{\text{tr}} = 10^{-8}$.

iii. **Integrand Construction:** Construct the integrand $f_1(x; a_1, \boldsymbol{\xi}_1)\, F_2(x; a_2, \boldsymbol{\xi}_2)$ using the OL-G PDF and CDF given in (2) and (1).
iv. **Numerical Integration:** Evaluate the integral using the adaptive numerical integration procedure implemented in R's `stats::integrate()`, with relative quadrature tolerance $\epsilon_{\text{quad}} = 10^{-8}$.
v. **Error Assessment:** Record the numerical integration error estimate returned by the quadrature procedure. When an unbounded support is truncated, the truncation contribution is additionally bounded by (12).

This direct numerical approach provides a general and numerically stable framework for evaluating reliability within the OL-G family without relying on the divergent series representation.

5 Numerical studies

This section presents numerical studies to assess the proposed direct numerical integration framework for reliability evaluation within the OL-G family under heterogeneous baseline distributions.

5.1 Simulation study and validation

In accordance with the definition adopted in Section 4, the reliability measure is $R = P(X_1 > X_2)$. For the Monte Carlo validation, we evaluate the complementary probability $1 - R = P(X_2 > X_1)$, which is estimated directly from simulated pairs. Thus, the Monte Carlo results reported below correspond to the complementary reliability measure, whereas the numerical integration provides the value of R.

We first consider a heterogeneous OL-G model in which the two variables have different baseline distributions:

- $X_1 \sim$ OL-Fréchet$(a_1 = 1.0, \sigma_1 = 1.0, \xi_1 = 2.0)$,
- $X_2 \sim$ OL-log-logistic$(a_2 = 1.0, \alpha_2 = 1.0, \beta_2 = 2.0)$.

The parameters are selected to provide substantial overlap between the two distributions. All calculations were performed in the statistical environment R (R, 4.3.0).

To validate the numerical integration, a Monte Carlo study was conducted. For each sample size $N = 100, 200, 300, 400, 600, 1000$, $M = 1000$ independent replications were performed. In each replication, N independent pairs (X_{1i}, X_{2i}) were generated, and the complementary reliability was estimated by the proportion of pairs in which X_{2i} exceeded X_{1i}. The overall Monte Carlo estimate was obtained by averaging these proportions over the M replications. The simulation results are summarized in Table 2.

Table 2: Monte Carlo simulation results for numerical integration ($M = 1000$).

n	Integral	MC mean	Bias	Rel. bias	Standard error
100	0.3816	0.3860	0.0044	1.1583%	0.0016
200	0.3816	0.3851	0.0035	0.9211%	0.0011
300	0.3816	0.3840	0.0024	0.6385%	0.0008
400	0.3816	0.3841	0.0025	0.6754%	0.0007
600	0.3816	0.3824	0.0008	0.2219%	0.0006
1000	0.3816	0.3834	0.0018	0.4921%	0.0005

The results show close agreement between the direct numerical integration and the Monte Carlo estimates of the complementary reliability $1 - R = P(X_2 > X_1)$. Across all sample sizes, the Monte Carlo means remain close to the numerical reference value. The absolute bias decreases from 0.0044 at $N = 100$ to 0.0008 at $N = 600$, while the small fluctuations observed at larger sample sizes are consistent with ordinary Monte Carlo sampling variability. The relative bias remains below 1.2% for all considered sample sizes and is below 0.5% at $N = 1000$. The standard errors also decrease with increasing sample size. These results provide numerical evidence that the proposed direct integration framework is consistent with independent Monte Carlo evaluation.

To provide an additional numerical assessment, the direct integration result was compared with two alternative approximations: a Laplace approximation and an asymptotic approximation based on the tail behavior of the integrand. The results are reported in Table 3.

Table 3: Comparison of numerical integration with analytical approximations.

Method	Value	Interpretation
Numerical integration	0.6162	$R = P(X_1 > X_2)$
Laplace approximation	0.6284	Approximation to R
Asymptotic approximation	0.3838	$1 - R = P(X_2 > X_1)$

The numerical integration gives $R = 0.6162$. The Laplace approximation gives 0.6284, which is within about 2% of the numerical result, indicating close agreement. The asymptotic approximation gives 0.3838, which corresponds exactly to the complementary probability $1 - R$, since $1 - 0.6162 = 0.3838$. Thus, the asymptotic value is consistent with the complementary reliability obtained from the numerical integration.

In addition to numerical accuracy, the computational efficiency of the proposed framework was examined for several heterogeneous baseline distribution pairs. The implementation uses vectorized function evaluation, quantile-based truncation for unbounded supports, and adaptive numerical quadrature. The corresponding computation times are reported in Table 4. Note that the times are medians of 10 independent runs. The computation times remain below 0.2 seconds for all considered scenarios, indicating that the proposed numerical framework is computationally efficient for the evaluated heterogeneous OL-G models.

Table 4: Computation time for different baseline distribution pairs.

Scenario	R	Time (seconds)
Fréchet vs Log-Logistic	0.6162	0.16
Weibull vs Log-Normal (overlap)	0.6374	0.16
Weibull vs Log-Normal (real data)	0.9223	0.02
Gamma vs Log-Normal	0.0248	0.14

5.2 Numerical example: A practical engineering scenario

Here, the reliability is defined as $R = P(\text{Strength} > \text{Stress})$, which is the probability that the material strength exceeds the applied stress. To illustrate our numerical framework, we consider an engineering scenario with two independent datasets. The strength data consist of 60 tensile strength measurements from structural steel specimens, modeled by a Weibull distribution. The stress data comprise 60 load measurements from infrastructure, modeled by a Log-normal distribution. This setup reflects a realistic case where strength and stress follow different distributions, a common situation that existing analytical formulas cannot handle. We fit OL-G distributions to each dataset using maximum likelihood estimation:

- **Strength data:** OL-Weibull distribution,
- **Stress data:** OL-lognormal distribution.

The choice of these distributions is justified by both physical principles and empirical evidence. The Kolmogorov-Smirnov tests confirm adequate fit for both distributions, with p-values of 0.9336 for strength and 0.5639 for stress, both exceeding the conventional 0.05 threshold. All analyses were based on 60 observations for each dataset. The estimated parameters for both OL-G models are presented in Table 5, and the reliability results are summarized in Table 6. Here, the parameters are estimated via ML method. In addition, the bootstrap confidence interval (CI) is obtained based on $B = 1000$ parametric bootstrap replications.

Table 5: Parameter estimates for OL-G models.

Parameter	OL-Weibull (Strength)	OL-lognormal (Stress)
a	0.5012	0.6352
Shape	19.3905	—
Scale	421.5014	—
Meanlog	—	5.5396
Sdlog	—	0.4749
AIC	561.2600	749.6800
KS test p-value	0.9336	0.5639

Table 6: Reliability assessment results.

Metric	Value
Empirical reliability (R_{emp})	0.9000
Numerical integration (R_{num})	0.9221
Absolute difference	0.0221
Relative error	2.47%
95% Bootstrap CI for R	(0.8543, 0.9778)

Figure 1 presents the Q-Q and P-P plots for both fitted distributions. The close alignment of the points with the reference lines visually confirms the adequacy of the fitted OL-G models. Together with the KS test results, these diagnostics support the claim that the slight discrepancy between R_{emp} and R_{num} (relative error of 2.47%) is attributable to sampling variability rather than model misspecification.

This numerical example demonstrates three critical points:

1. **Limitation of existing methods:** The analytical formula proposed by Gomes-Silva et al. (2017) fails in practical engineering scenarios where strength and stress variables follow different baseline distributions.
2. **General applicability:** The proposed numerical framework successfully handles heterogeneous distributions, computing reliability with acceptable accuracy (2.47% relative error). The slight difference between R_{emp} and R_{num} arises because R_{emp} is calculated directly from the raw data, while R_{num} is based on the fitted OL-G distributions. The bootstrap confidence interval provides additional insight into the uncertainty of

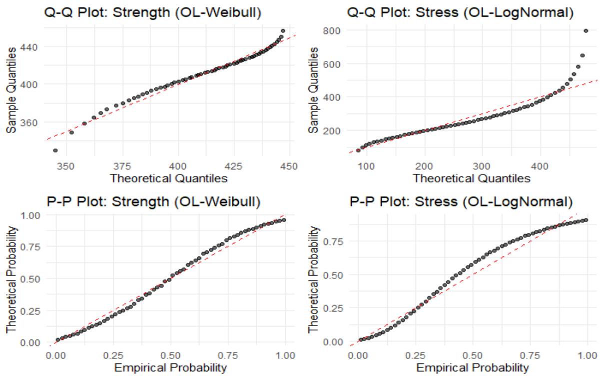

Figure 1: Q-Q plots and P-P plots for the fitted OL-Weibull (strength) and OL-lognormal (stress) distributions. The plots confirm that both fitted distributions adequately capture the empirical characteristics of the data, as the points closely follow the reference lines.

this estimate.

3. **Practical utility:** The framework provides engineers with a practical tool for reliability assessment in real-world applications where distributional homogeneity cannot be assumed.

6 Conclusion

This paper presents a generalized numerical integration framework for computing stress-strength reliability within the OL-G family of distributions. We demonstrate that the existing series expansion method proposed by Gomes-Silva et al. (2017) diverges in the homogeneous-baseline setting, rendering it impractical for reliability evaluation. The proposed numerical approach overcomes this limitation by directly evaluating the reliability integral and accommodating distinct baseline distributions without relying on the divergent series representation. Extensive simulations and a practical engineering example involving OL-Weibull and OL-lognormal distributions confirm the accuracy, stability, and applicability of the method. The framework successfully handles scenarios where stress and strength arise from different physical processes, bridging a critical gap between theoretical models and practical reliability analysis.

Limitations

The proposed framework has several limitations. First, numerical integration over a truncated domain $[L, U]$, with $L = F_1^{-1}(\epsilon)$ and $U = F_1^{-1}(1-\epsilon)$, where F_1 is the CDF of X_1, introduces a trade-off between truncation error and computational cost. The parameter ϵ controls this balance: smaller values increase precision but require more quadrature nodes. Second, computational complexity grows with the parametric dimension of the baseline distribution. While the models considered here are low-dimensional, the framework may become costly for high-dimensional or hierarchical baselines. Third, the reliability estimate is model-dependent. Misspecification of the baseline distribution—whether due to incorrect form or poor estimation—directly biases the estimate, especially for small samples or non-standard data features such as multimodality or heavy tails. Fourth, the current framework assumes complete data and does not support censoring. Finally, the empirical results are limited to a specific set of baseline distributions. Further investigation is needed to assess generalizability across the broader OL-G family.

Future directions

Several promising directions can extend the proposed framework. Dependence between stress and strength variables may be modeled via copulas, with the Clayton copula being a suitable choice for lower-tail dependence; however, this introduces a two-dimensional integral requiring adaptive cubature methods. A fully Bayesian approach using Hamiltonian Monte Carlo could provide credible intervals for the reliability measure while accounting for parameter uncertainty. The framework can also be adapted to right-censored data through modifications to the likelihood function and extended to multicomponent systems such as k-out-of-n structures. A dedicated R package would further enhance the practical accessibility of the methodology for practitioners and researchers.

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