Journal of Statistical Modelling: Theory and Applications
Vol. 7, No. 1, 2026, pp. 187-218
Yazd University Press 2026

JSMTA

Research Paper

Constant-stress partially accelerated life testing for Weibull-half-logistic distribution under Type-I and Type-II censoring schemes

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Received: July 16, 2026/ Revised: October 21, 2026/ Accepted: October 31, 2026

Abstract: The constant-stress partially accelerated life tests under Type-I and Type-II censoring schemes are considered when the lifetime of the units follows a Weibull-half-logistic distribution. Estimation techniques for the distribution parameters and the acceleration factor are discussed, including maximum likelihood estimates, Bayesian, and E-Bayesian estimates. Since there are no closed-form solutions for maximum likelihood estimates, we resort to establishing approximate confidence intervals for the parameters using the Fisher information matrices and also provide the bootstrap confidence intervals. The Bayesian estimates under the squared error, linear exponential, and general entropy loss functions are generated. Using the Markov chain Monte Carlo method and Lindley's approximation, the parameter estimates are approximated. Also, D-optimality and A-optimality criteria are used to find the optimal proportion of sample units allocated to the accelerated condition. Additionally, a real-life dataset is examined to highlight the practical utility of our methodologies.

Keywords: Bayesian estimates; Censoring; Constant-stress partially accelerated life test; Maximum likelihood estimate; Weibull-half-logistic distribution.
Mathematics Subject Classification (2010): 62F10, 62F15, 62N02.

1 Introduction

It may be difficult to obtain information during testing under normal conditions, as lifetime testing is costly and time-consuming. Statisticians seek information about the lifetimes of products and materials to improve and develop them. So, it is preferable to

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use accelerated life tests (ALTs) to assess failure data in the shortest possible time in fields such as manufacturing industries. In ALTs, test items are subjected to higher than normal stresses. Usually, these stresses are temperature, vibration, voltage, pressure, etc., which can cause premature failure. Therefore, units that have not failed are at risk.

One can divide ALTs into two types, the first one is fully ALTs (FALTs) which are based on the major assumption that the relationship between life and stress is known while the second one is partially ALTs (PALTs) in which the previous relationship is unknown or cannot be assumed. In accordance with Nelson (1990), FALTs are divided mainly into three types. The first one is constant-stress ALT (CSALT) which is the most common type of stress. In this type, the sample items are exposed to continuous stress until failure or censoring whichever occurs first. When there is a large variation in failure times, constant-stress may take too much time. So, to ensure failure occurs faster, an alternative approach is needed. As a result, step-stress ALT (SSALT) appeared to overcome this obstacle and to be more efficient and practical than constant stress. Under this type, the test item is exposed to a level of stress for a pre-specified period until it fails but if it does not fail, the level of stress to which it is exposed is raised and increases repeatedly until the test item fails or the censored condition is reached. The third type is considered progressive-stress ALT in which the test items are exposed to continuously increasing stress with time. Sometimes, the data of FALTs cannot be extrapolated to normal use conditions because the nature of the life-stress relationship is unknown, so partially accelerated life tests (PALTs) are a good alternative to use in such cases in estimating the acceleration factor. Like FALTs, PALTs are also mainly divided into three types.

The first one is CSPALT where the sample items are run at either normal or accelerated conditions, i.e. each test item is run at a constant stress level until the test is terminated. The second one, SSPALT, the test item is firstly run at normal stress with a pre-specified time (stress change time) until failing but if it does not fail, the test condition is switched to a higher stress level in which the item is exposed to steady stress until failure occurs or censoring reaches, that is, the total lifetime of the test item goes through two stages, normal use condition and accelerated condition, respectively. Again, the third type is PSPALT, see Fathi et al. (2024) and references therein for some statistical distributions under different sampling schemes.

In this paper, we focus on CSPALT and Weibull-half-logistic distribution (WHLD), which have not been investigated under any type of censoring scheme so far. The article is organized as follows: Section 2 introduces the Weibull-G family of distributions, and the CSPALT. Type-I and Type-II censoring schemes and their definitions and notations are given in this Section. The estimation techniques for the parameters of WHLD and also the acceleration factor are discussed. In Sections 3, the maximum likelihood estimates (MLEs) and approximate confidence intervals (ACIs) are discussed. In Section 4, the bootstrap confidence intervals are provided to overcome the issue of constructing ACIs for the estimates. In Section 5, the Bayesian and E-Bayesian estimates are considered. Utilizing the squared error (SE), linear exponential (LINEX), and general entropy (GE) loss functions. For deriving the estimates, the MCMC method and Lindley's approximation are applied. Two optimality criteria (D-optimality and A-optimality) are used to find the optimal proportion of sample units allocated to accelerated condition

in Section 6. The simulation is carried out to demonstrate the proposed estimation techniques in Section 7. Real-world data is also provided for illustration in Section 8.

2 Preliminary concepts and definitions

2.1 Weibull-half-logistic distribution

The Weibull distribution is the most important distribution for problems in reliability and has been extensively used for modeling data in reliability, engineering and biological studies. Various generalizations have been introduced for the Weibull distribution. One of the generalizations is the Weibull-G family introduced by Bourguignon et al. (2014). Here we focus on one of the most important two-parameter distributions in this family, the WHLD. Exactly, the cumulative distribution function (CDF) of the WHLD is defined by

$$F_X(x|a,c) = 1 - e^{-a\left(\frac{e^x-1}{2}\right)^c}, \qquad x,a,c>0.$$

Thus, WHLD with scale parameter a and shape parameter c, respectively has the probability density function (PDF), and failure rate function as follows:

$$\begin{aligned} f_X(x|a,c) &= \frac{ac}{2}\left(\frac{e^x-1}{2}\right)^{c-1} e^{x-a\left(\frac{e^x-1}{2}\right)^c}, \\ h_X(x|a,c) &= \frac{ac}{2}e^x\left(\frac{e^x-1}{2}\right)^{c-1}, \qquad x,a,c>0, \end{aligned}$$

respectively. The probability density and failure rate functions of WHLD are given in Figure 1 for some values of parameters. The failure rate function of WHLD is an increasing ($c>1$), the constant ($c=1$) or decreasing ($0<c<1$) functions for different values of the parameters. Therefore, if empirical analyses indicate that the failure rate function of the prior distribution is bathtub-shaped, it can be used to analyze real-world datasets. Also, it is observed that the PDF of WHLD is either decreasing or unimodal (Kazemi and Kohansal, 2021; Kazemi and Zamani Mehreyan, 2024).

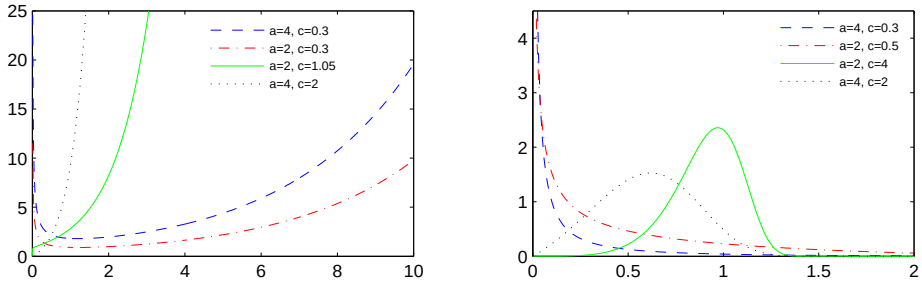

Figure 1: Shape of failure rate (left) and probability density (right) functions of WHLD (Kazemi and Kohansal, 2021).

The most important members of Weibull-G family are Weibull-exponential, Weibull-Weibull, Weibull-Frechet, Weibull-power function, Weibull-Pareto, Weibull-Burr XII,

Weibull-log-logistic, Weibull-Lomax, Weibull-Gumbel, Weibull-Kumaraswamy and Weibull-normal distributions. These distributions have been used in various fields and have shown their superiority over other distributions for analyzing various data. Therefore, the analysis of this paper is valid with proportional changes for all two-parameter distributions belonging to this family. For the above distributions, we will have to calculate more partial calculations and again the analysis performed is valid.

2.2 Constant-stress partially accelerated life tes

In a CSPALT, all of the n items are divided into two groups. In fact, m items are randomly chosen from the n items and are allocated to accelerated conditions. Thus, the remaining $n-m$ items are placed to normal use conditions. The following assumptions are made in a CSPALT (Nelson, 1990):

- Each test item is run until the censoring time τ.
- The test condition is not changed.
- The lifetimes X_i , $i = m+1, \ldots, n$ and Y_j , $j = 1, \ldots, m$ of items allocated at normal use conditions and accelerated conditions, respectively, are i.i.d. random variables.
- The lifetimes X_i and Y_j are mutually independent.
- In fact $m = n\pi$ where π is the proportion of the sample items allocated to accelerated conditions.

In a CSPALT, we assume $Y = X/\beta$ where $\beta > 1$ is the acceleration factor which is the ratio of mean lifetime at use conditions to that at accelerated conditions. Then, for an item tested at acceleration conditions, we have

$$\begin{aligned} F_Y(y|a,c,\beta) &= 1 - e^{-a\left(\frac{e^{\beta y}-1}{2}\right)^c}, \\ f_Y(y|a,c,\beta) &= \frac{\beta a c}{2}\left(\frac{e^{\beta y}-1}{2}\right)^{c-1} e^{\beta y - a\left(\frac{e^{\beta y}-1}{2}\right)^c}, \qquad y > 0\ , a, c > 0. \end{aligned}$$

2.3 Type-I and Type-II censoring schemes

There are many cases where failure may not occur in a sample collected from lifetime data. Due to time or cost considerations, the practitioners will terminate the testing and report the results before all items realize their failures. In a Type-I censoring, the failure is observed only if it occurs prior to some prespecified time. Thus, all censored items have times equal to the length of the study period τ. With this description, censored data are usually obtained when censoring time is fixed. Then the number of failures in that fixed time is a random variable. Let δ_{ui} for $i = m+1, \ldots, n$ and δ_{wj} for $j = 1, \ldots, m$ denote the failure indicators such that

$$\delta_{ui} = \begin{cases} 1, & x_i \le \tau, \\ 0, & o.w. \end{cases} \qquad \delta_{wj} = \begin{cases} 1, & y_j \le \tau, \\ 0, & o.w. \end{cases}$$

Let c_u and c_w be the number of items censored at normal and accelerated conditions respectively. Then, if n_u and n_w are the number of items failed at normal and accelerated conditions, respectively: $\sum_{i=m+1}^{n} \delta_{ui} = n_u$ and $\sum_{j=1}^{m} \delta_{wj} = n_w$, and also

$c_u = n - m - n_u$ and $c_w = m - n_w$. Based on Type-I censoring scheme, the likelihood function for $\{(x_i, \delta_{ui}), (y_j, \delta_{wj}) : i = m+1, \ldots, n, j = 1, \ldots, m\}$ is given by

$$L_1(a, c, \beta) = L_{1u}(x_i, \delta_{ui}|a, c) \times L_{1w}(y_j, \delta_{wj}|a, c, \beta), \tag{1}$$

where

$$\begin{aligned} L_{1u}(x_i, \delta_{ui}|a, c) &= \prod_{i=m+1}^{n} f_X(x_i|a, c)^{\delta_{ui}} (1 - F_X(\tau|a, c))^{1-\delta_{ui}}, \\ L_{1w}(y_j, \delta_{wj}|a, c, \beta) &= \prod_{j=1}^{m} f_Y(y_j|a, c, \beta)^{\delta_{wj}} (1 - F_Y(\tau|a, c, \beta))^{1-\delta_{wj}}. \end{aligned}$$

In testing of equipments, another censoring scheme often used is Type-II censoring where all items are put on test at the same time. In this scheme, the test is ended when the predetermined r of the items have failed. Because of the test could take a very long time for all items to fail, thus this experiment saves time and money. For a CSPALT with Type-II censoring, the failure times consist of rth smallest lifetimes $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(r)}$ out of a random sample of $n - m$ lifetimes $X_{m+1}, \ldots, X_n$ under normal use conditions and $y_{(1)} \leq y_{(2)} \leq \ldots \leq y_{(r)}$ out of a random sample of m lifetimes $Y_1, \ldots, Y_m$ at accelerated conditions, respectively. Let δ_{ui} for $i = m+1, \ldots, n$ and δ_{wj} for $j = 1, \ldots, m$ denote the failure indicators such that

$$\delta_{ui} = \begin{cases} 1, & x_i \leq x_{(r)}, \\ 0, & o.w. \end{cases} \qquad \delta_{wj} = \begin{cases} 1, & y_j \leq y_{(r)}, \\ 0, & o.w. \end{cases}$$

In this case, $c_u = n - m - r$ and $c_w = m - r$. Based on Type-II censoring scheme, the likelihood function for $\{(x_i, \delta_{ui}), (y_j, \delta_{wj}) : i = m+1, \ldots, n, j = 1, \ldots, m\}$ is given by

$$L_2(a, c, \beta) = L_{2u}(x_i, \delta_{ui}|a, c) \times L_{2w}(y_j, \delta_{wj}|a, c, \beta), \tag{2}$$

where

$$\begin{aligned} L_{2u}(x_i, \delta_{ui}|a, c) &= \frac{(n-m)!}{(n-m-r)!} \prod_{i=1}^{r} f_X(x_{(i)}|a, c)(1 - F_X(x_{(r)}|a, c))^{n-m-r}, \\ L_{2w}(y_j, \delta_{wj}|a, c, \beta) &= \frac{m!}{(m-r)!} \prod_{j=1}^{r} f_Y(y_{(j)}|a, c)(1 - F_Y(y_{(r)}|a, c))^{m-r}. \end{aligned}$$

3 Maximum likelihood estimation and approximate confidence interval

3.1 Inference under Type-I censoring scheme

From (1),

$$L_1(a, c, \beta) = \left(\frac{ac}{2}\right)^{n_u+n_w} \beta^{n_w} \prod_{i=m+1}^{n} \left(\frac{e^{x_i} - 1}{2}\right)^{\delta_{ui}(c-1)} e^{\sum_{i=m+1}^{n} \delta_{ui} \left(x_i - a2^{-c}(e^{x_i}-1)^c\right)}$$

$$\times \prod_{i=1}^{m} \left(\frac{e^{\beta y_i} - 1}{2} \right)^{\delta_{wi}(c-1)} e^{\sum_{i=1}^{m} \delta_{wi} \left(\beta y_i - a 2^{-c} (e^{\beta y_i - 1})^c \right)}$$
$$\times e^{-ac_u \left(\frac{e^{\tau}-1}{2} \right)^c - ac_w \left(\frac{e^{\beta\tau}-1}{2} \right)^c}. \tag{3}$$

The natural logarithm of (3), the log-likelihood function, is

$$\begin{aligned}
\ell_1 &= \log L_1(a, c, \beta) \\
&= (n_u + n_w)(\log a + \log c - c \log 2) + n_w \log \beta \\
&\quad + \sum_{i=m+1}^{n} \delta_{ui} \left((c-1) \log(e^{x_i} - 1) + x_i - a 2^{-c} (e^{x_i} - 1)^c \right) - a c_u 2^{-c} (e^{\tau} - 1)^c \\
&\quad + \sum_{i=1}^{m} \delta_{wi} \left((c-1) \log(e^{\beta y_i} - 1) + \beta y_i - a 2^{-c} (e^{\beta y_i} - 1)^c \right) - a c_w 2^{-c} (e^{\beta\tau} - 1)^c.
\end{aligned}$$

The MLEs of the parameters a, c and β are solutions to the system of likelihood equations obtained by

$$\begin{aligned}
\frac{\partial \ell_1}{\partial a} &= \frac{n_u + n_w}{a} - \sum_{i=m+1}^{n} \delta_{ui} 2^{-c} (e^{x_i} - 1)^c - c_u 2^{-c} (e^{\tau} - 1)^c \\
&\quad - \sum_{i=1}^{m} \delta_{wi} 2^{-c} (e^{\beta y_i} - 1)^c - c_w 2^{-c} (e^{\beta\tau} - 1)^c = 0, \\
\frac{\partial \ell_1}{\partial c} &= \left(\frac{1}{c} - \log 2 \right) (n_u + n_w) + \sum_{i=m+1}^{n} \delta_{ui} \left[\log(e^{x_i} - 1) - a \left(\frac{e^{x_i} - 1}{2} \right)^c \log \left(\frac{e^{x_i} - 1}{2} \right) \right] \\
&\quad + \sum_{i=1}^{m} \delta_{wi} \left[\log(e^{\beta y_i} - 1) - a \left(\frac{e^{\beta y_i} - 1}{2} \right)^c \log \left(\frac{e^{\beta y_i} - 1}{2} \right) \right] \\
&\quad - c_u a \left(\frac{e^{\tau} - 1}{2} \right)^c \log \left(\frac{e^{\tau} - 1}{2} \right) - c_w a \left(\frac{e^{\beta\tau} - 1}{2} \right)^c \log \left(\frac{e^{\beta\tau} - 1}{2} \right) = 0, \\
\frac{\partial \ell_1}{\partial \beta} &= \frac{n_w}{\beta} + \sum_{i=1}^{m} \delta_{wi} \left[\frac{(c-1) y_i e^{\beta y_i}}{e^{\beta y_i} - 1} + y_i - ac 2^{-c} y_i e^{\beta y_i} (e^{\beta y_i} - 1)^{c-1} \right] \\
&\quad - c_w ac 2^{-c} \tau e^{\beta\tau} (e^{\beta\tau} - 1)^{c-1} = 0.
\end{aligned}$$

The MLEs of a, c and β symbolized by $\hat{a}^1, \hat{c}^1$ and $\hat{\beta}^1$, are then derived from the above equations. Due to the challenge of obtaining the MLEs, we utilize an iterative method such as the Newton-Raphson method to solve the equations.

Because there are no closed-form solutions available for the MLEs, the precise bounds of the confidence intervals (CIs) for these parameters are impossible to be derived. Hence, we resort to establishing approximate confidence intervals (ACIs) for the three parameters using the Fisher information matrix. Suppose $\boldsymbol{\theta} = (a, c, \beta)$, then the asymptotic covariance matrix denoted as $\mathcal{F}_1^{-1}(\boldsymbol{\theta})$ is utilized to obtain the

approximate CIs:

$$\mathcal{F}_1^{-1}(\boldsymbol{\theta}) = \begin{bmatrix} -I_{11}^{(1)} & -I_{12}^{(1)} & -I_{13}^{(1)} \\ -I_{21}^{(1)} & -I_{22}^{(1)} & -I_{23}^{(1)} \\ -I_{31}^{(1)} & -I_{32}^{(1)} & -I_{33}^{(1)} \end{bmatrix}^{-1} \approx \begin{bmatrix} Var(\hat{a}^1) & Cov(\hat{a}^1, \hat{c}^1) & Cov(\hat{a}^1, \hat{\beta}^1) \\ Cov(\hat{c}^1, \hat{a}^1) & Var(\hat{c}^1) & Cov(\hat{c}^1, \hat{\beta}^1) \\ Cov(\hat{\beta}^1, \hat{a}^1) & Cov(\hat{\beta}^1, \hat{c}^1) & Var(\hat{\beta}^1) \end{bmatrix}.$$

The elements of the observed Fisher information matrix are presented in Appendix A. So, $\hat{\boldsymbol{\theta}}^1 \sim N(\boldsymbol{\theta}, \hat{\mathcal{F}}_1^{-1}(\boldsymbol{\theta}))$. Now, suppose z_α is the 100α upper percentage point of the standard normal distribution. Thus, an asymptotic $(1-\alpha)\times 100\%$ confidence intervals for $\boldsymbol{\theta}$ is given by

$$\left(\hat{\boldsymbol{\theta}}^1 - z_{\frac{\alpha}{2}}\sqrt{\widehat{Var(\hat{\boldsymbol{\theta}}^1)}}, \hat{\boldsymbol{\theta}}^1 + z_{\frac{\alpha}{2}}\sqrt{\widehat{Var(\hat{\boldsymbol{\theta}}^1)}}\right).$$

Occasionally, the ACIs yield a negative lower bound even though the parameters are strictly non-negative. To conquer this obstacle, the delta method (William, 2000) and the logarithmic transformation (Meeker and Escobar, 1998) can be applied. Hence, the asymptotic distribution of $\ln \hat{\boldsymbol{\theta}}^{\mathbf{1}}$ is $\log \hat{\boldsymbol{\theta}}^1 - \log \boldsymbol{\theta} \xrightarrow{d} N(\mathbf{0}, Var(\log \hat{\boldsymbol{\theta}}^1))$, where $Var(\log \hat{\boldsymbol{\theta}}^1) = \frac{\widehat{Var(\hat{\boldsymbol{\theta}}^1)}}{\hat{\boldsymbol{\theta}}^{1^2}}$. Hence, the ACIs based on log-transformed MLEs are

$$\left(\hat{\boldsymbol{\theta}}^1 \times \exp\left(-\frac{z_{\frac{\alpha}{2}}}{\hat{\boldsymbol{\theta}}^1}\sqrt{\widehat{Var(\hat{\boldsymbol{\theta}}^1)}}\right), \hat{\boldsymbol{\theta}}^1 \times \exp\left(\frac{z_{\frac{\alpha}{2}}}{\hat{\boldsymbol{\theta}}^1}\sqrt{\widehat{Var(\hat{\boldsymbol{\theta}}^1)}}\right)\right).$$

3.2 Inference under Type-II censoring scheme

It should be noted that in this case $c_u = n - m - r$ and $c_w = m - r$. From (2),

$$\begin{aligned} L_2(a,c,\beta) &= \left(\frac{ac}{2}\right)^{2r} \beta^r \prod_{i=1}^{r}\left(\frac{e^{x_{(i)}}-1}{2}\right)^{c-1} e^{\sum_{i=1}^{r}\left(x_{(i)} - a2^{-c}(e^{x_{(i)}}-1)^c\right)} \\ &\quad \times \prod_{j=1}^{r}\left(\frac{e^{\beta y_{(j)}}-1}{2}\right)^{c-1} e^{\sum_{j=1}^{r}\left(\beta y_{(j)} - a2^{-c}(e^{\beta y_{(j)}}-1)^c\right)} \\ &\quad \times e^{-ac_u\left(\frac{e^{x_{(r)}}-1}{2}\right)^c - ac_w\left(\frac{e^{\beta y_{(r)}}-1}{2}\right)^c}. \end{aligned} \tag{4}$$

Then

$$\begin{aligned} \ell_2 = \log L_2(a,c,\beta) &= 2r(\log a + \log c - c\log 2) + r\log\beta \\ &+ \sum_{i=1}^{r}\left((c-1)\log(e^{x_{(i)}}-1) + x_{(i)} - a2^{-c}(e^{x_{(i)}}-1)^c\right) - ac_u 2^{-c}(e^{x_{(r)}}-1)^c \\ &+ \sum_{j=1}^{r}\left((c-1)\log(e^{\beta y_{(j)}}-1) + \beta y_{(j)} - a2^{-c}(e^{\beta y_{(j)}}-1)^c\right) - ac_w 2^{-c}(e^{\beta y_{(r)}}-1)^c. \end{aligned}$$

The MLEs of the parameters a, c and β are solutions to the system of likelihood equations obtained by

$$\frac{\partial \ell_2}{\partial a} = \frac{2r}{a} - \sum_{i=1}^{r} 2^{-c}(e^{x_{(i)}}-1)^c - c_u 2^{-c}(e^{x_{(r)}}-1)^c$$

$$
\begin{aligned}
&-\sum_{j=1}^{r} 2^{-c}(e^{\beta y_{(j)}}-1)^c - c_w 2^{-c}(e^{\beta y_{(r)}}-1)^c = 0,\\
\frac{\partial \ell_2}{\partial c} &= 2r\left(\frac{1}{c}-\log 2\right)+\sum_{i=1}^{r}\left[\log(e^{x_{(i)}}-1)-a\left(\frac{e^{x_{(i)}}-1}{2}\right)^c \log\left(\frac{e^{x_{(i)}}-1}{2}\right)\right]\\
&\quad -c_u a\left(\frac{e^{x_{(r)}}-1}{2}\right)^c \log\left(\frac{e^{x_{(r)}}-1}{2}\right)\\
&\quad +\sum_{j=1}^{r}\left[\log(e^{\beta y_{(j)}}-1)-a\left(\frac{e^{\beta y_{(j)}}-1}{2}\right)^c \log\left(\frac{e^{\beta y_{(j)}}-1}{2}\right)\right]\\
&\quad -c_w a\left(\frac{e^{\beta y_{(r)}}-1}{2}\right)^c \log\left(\frac{e^{\beta y_{(r)}}-1}{2}\right)=0,\\
\frac{\partial \ell_2}{\partial \beta} &= \frac{r}{\beta}+\sum_{j=1}^{r}\left[\frac{(c-1)y_{(j)}e^{\beta y_i}}{e^{\beta y_i}-1}+y_{(j)}-ac2^{-c}y_{(j)}e^{\beta y_{(j)}}(e^{\beta y_{(j)}}-1)^{c-1}\right]\\
&\quad -c_w ac2^{-c}y_{(r)}e^{\beta y_{(r)}}(e^{\beta y_{(r)}}-1)^{c-1}=0.
\end{aligned}
$$

The estimates of a, c and β symbolized by $\hat{a}^2, \hat{c}^2$ and $\hat{\beta}^2$, are then derived from the above equations. Again, we utilize the Newton-Raphson method to obtain the MLEs and then the asymptotic covariance matrix denoted as $\mathcal{F}_2^{-1}(\boldsymbol{\theta})$ is utilized to obtain the approximate CIs:

$$
\mathcal{F}_2^{-1}(\boldsymbol{\theta})=\begin{bmatrix} -I_{11}^{(2)} & -I_{12}^{(2)} & -I_{13}^{(2)}\\ -I_{21}^{(2)} & -I_{22}^{(2)} & -I_{23}^{(2)}\\ -I_{31}^{(2)} & -I_{32}^{(2)} & -I_{33}^{(2)}\end{bmatrix}^{-1} \approx \begin{bmatrix} Var(\hat{a}^2) & Cov(\hat{a}^2,\hat{c}^2) & Cov(\hat{a}^2,\hat{\beta}^2)\\ Cov(\hat{c}^2,\hat{a}^2) & Var(\hat{c}^2) & Cov(\hat{c}^2,\hat{\beta}^2)\\ Cov(\hat{\beta}^2,\hat{a}^2) & Cov(\hat{\beta}^2,\hat{c}^2) & Var(\hat{\beta}^2)\end{bmatrix}.
$$

The elements of the observed Fisher information matrix are presented in Appendix B. So, $\hat{\boldsymbol{\theta}}^2 \sim N(\boldsymbol{\theta}, \hat{\mathcal{F}}_2^{-1}(\boldsymbol{\theta}))$ and then

$$
\left(\hat{\boldsymbol{\theta}}^2 - z_{\frac{\alpha}{2}}\sqrt{\widehat{Var(\hat{\boldsymbol{\theta}}^2)}}, \hat{\boldsymbol{\theta}}^2 + z_{\frac{\alpha}{2}}\sqrt{\widehat{Var(\hat{\boldsymbol{\theta}}^2)}}\right).
$$

In this case, $\log\hat{\boldsymbol{\theta}}^2 - \log\boldsymbol{\theta} \xrightarrow{d} N(\mathbf{0}, Var(\log\hat{\boldsymbol{\theta}}^2))$, where $Var(\log\hat{\boldsymbol{\theta}}^2) = \frac{\widehat{Var(\hat{\boldsymbol{\theta}}^2)}}{\hat{\boldsymbol{\theta}}^{2^2}}$. Hence, the ACIs based on log-transformed MLEs are

$$
\left(\hat{\boldsymbol{\theta}}^2 \times \exp\left(-\frac{z_{\frac{\alpha}{2}}}{\hat{\boldsymbol{\theta}}^2}\sqrt{\widehat{Var(\hat{\boldsymbol{\theta}}^2)}}\right), \hat{\boldsymbol{\theta}}^2 \times \exp\left(\frac{z_{\frac{\alpha}{2}}}{\hat{\boldsymbol{\theta}}^2}\sqrt{\widehat{Var(\hat{\boldsymbol{\theta}}^2)}}\right)\right).
$$

4 Bootstrap confidence intervals (BCIs)

If the sample size is not large enough, the accuracy and efficiency of the normal approximation of MLEs may decrease. Therefore, the bootstrap confidence intervals are provided to overcome the issue of constructing ACIs. As mentioned earlier, the CIs based on the asymptotic results may not perform well. On the other hand, traditional statistical methods may struggle with small sample sizes. Therefore parametric bootstrap by resampling from the estimated parametric distribution of the data, allow the

generation of a large number of bootstrap samples. This process provides an estimate of the sampling distribution of a statistic of interest. It means that especially when dealing with small samples, CIs constructed using parametric bootstrap is more reliable and accurate. Two parametric bootstrap techniques are: the bootstrap percentile confidence interval (Boot-p) and bootstrap-t confidence interval (Boot-t), see Zhang and Gui (2022).

4.1 Parametric Boot-p

1. Utilize the Type-I censored sample to compute the MLEs $(\hat{a}^1, \hat{c}^1, \hat{\beta}^1)$ and Type-II censored sample to compute the MLEs $(\hat{a}^2, \hat{c}^2, \hat{\beta}^2)$.
2. Generate a Type-I censored sample and Type-II censored sample by regarding $(\hat{a}^1, \hat{c}^1, \hat{\beta}^1)$ and $(\hat{a}^2, \hat{c}^2, \hat{\beta}^2)$ as the parameter values for the WHLD, respectively.
3. Compute the MLEs $(\hat{a}^{1*}, \hat{c}^{1*}, \hat{\beta}^{1*})$ and $(\hat{a}^{2*}, \hat{c}^{2*}, \hat{\beta}^{2*})$ using the samples from Step 2.
4. Repeat Step 2 and 3, T times.
5. Estimate the CDF of $\hat{\theta}^{1*}$ by $\hat{F}_1(x) = P(\hat{\theta}^{1*} \leq x)$ and CDF of $\hat{\theta}^{2*}$ by $\hat{F}_2(x) = P(\hat{\theta}^{2*} \leq x)$ where $\theta = a, c, \beta$ and let $\hat{\theta}^1_{Boot-p} = \hat{F}_1^{-1}(x)$ and $\hat{\theta}^2_{Boot-p} = \hat{F}_2^{-1}(x)$ for a given $0 < x < 1$.

Then, the $100(1-\alpha)\%$ boot-p confidence interval of θ are given by

$$\begin{aligned}(\hat{\theta}^1_l, \hat{\theta}^1_u) &= \Big(\hat{\theta}^1_{Boot-p}\Big(\frac{\alpha}{2}\Big), \hat{\theta}^1_{Boot-p}\Big(1-\frac{\alpha}{2}\Big)\Big),\\ (\hat{\theta}^2_l, \hat{\theta}^2_u) &= \Big(\hat{\theta}^2_{Boot-p}\Big(\frac{\alpha}{2}\Big), \hat{\theta}^2_{Boot-p}\Big(1-\frac{\alpha}{2}\Big)\Big).\end{aligned}$$

4.2 Parametric Boot-t

1. Step 1 is the same with the boot-p confidence interval algorithm.
2. Step 2 is the same with the boot-p confidence interval algorithm.
3. Compute the MLEs $(\hat{a}^{1*}, \hat{c}^{1*}, \hat{\beta}^{1*})$ and $(\hat{a}^{2*}, \hat{c}^{2*}, \hat{\beta}^{2*})$ using the samples from Step 2 and obtain the statistics: $T_1^* = (\hat{\theta}^{1*} - \hat{\theta}^1)/\sqrt{\widehat{Var}(\hat{\theta}^{1*})}$, $T_2^* = (\hat{\theta}^{2*} - \hat{\theta}^2)/\sqrt{\widehat{Var}(\hat{\theta}^{2*})}$.
4. Repeat Step 2 and 3, T times.
5. Estimate the CDF of T_1^* by $\hat{F}_1(x) = P(T_1^* \leq x)$ and CDF of T_2^* by $\hat{F}_2(x) = P(T_2^* \leq x)$ where $\theta = a, c, \beta$ and let $\hat{\theta}_{Boot-t} = \hat{\theta}^1 + \sqrt{\widehat{Var}(\hat{\theta}^{1*})}\hat{F}_1^{-1}(x)$ and $\tilde{\theta}_{Boot-t} = \hat{\theta}^2 + \sqrt{\widehat{Var}(\hat{\theta}^{2*})}\hat{F}_2^{-1}(x)$ for a given $0 < x < 1$.

Then, the $100(1-\alpha)\%$ boot-t confidence interval of θ are given by

$$\begin{aligned}(\hat{\theta}^1_l, \hat{\theta}^1_u) &= \Big(\hat{\theta}^1_{Boot-t}\Big(\frac{\alpha}{2}\Big), \hat{\theta}^1_{Boot-t}\Big(1-\frac{\alpha}{2}\Big)\Big),\\ (\hat{\theta}^2_l, \hat{\theta}^2_u) &= \Big(\hat{\theta}^2_{Boot-t}\Big(\frac{\alpha}{2}\Big), \hat{\theta}^2_{Boot-t}\Big(1-\frac{\alpha}{2}\Big)\Big).\end{aligned}$$

5 Bayes estimations

5.1 Symmetric and asymmetric loss functions

The first step is to determine the appropriate prior distributions for the unknown parameters a, c and β (Zellner, 1986). Due to more flexibility in covering a large variety of prior beliefs and the strictly positive support of a, c and $\beta - 1$, we suppose

$$\begin{aligned} &\pi_1(a) \propto a^{\alpha_1 - 1} \exp(-\gamma_1 a), \qquad a, \alpha_1, \gamma_1 > 0, \\ &\pi_2(c) \propto c^{\alpha_2 - 1} \exp(-\gamma_2 c), \qquad c, \alpha_2, \gamma_2 > 0, \\ &\pi_3(\beta) \propto \beta^{-1}, \qquad \beta > 1, \end{aligned} \tag{5}$$

where $\alpha_1, \gamma_1, \alpha_2, \gamma_2$ are hyper-parameters. We assume the priors are independent and the prior for β is non-informative. Then, the joint prior density $\pi(a, c, \beta)$ is shown below:

$$\pi(a, c, \beta) \propto a^{\alpha_1 - 1} c^{\alpha_2 - 1} \beta^{-1} \exp\big(- (\gamma_1 a + \gamma_2 c)\Big). \tag{6}$$

Consequently, the joint posterior density can be formulated as follows:

$$\pi_s(a, c, \beta | \text{data}) = \frac{1}{K} L_s(a, c, \beta | \text{data}) \pi(a, c, \beta), \quad s = 1, 2, \tag{7}$$

where

$$K = \int_1^\infty \int_0^\infty \int_0^\infty L_s(a, c, \beta | \text{data}) \pi(a, c, \beta) da dc d\beta.$$

If at least one unit is allocated to the accelerated condition ($m > 0$), then the posterior density given in (7) is proper. It is sufficient to prove that $K < \infty$. Substituting the likelihood under Type-I censoring into the prior, the posterior kernel becomes

$$\pi(a, c, \beta \mid \text{data}) \propto a^{n_u + n_w + \alpha_1 - 1} c^{n_u + n_w + \alpha_2 - 1} \beta^{n_w - 1} Q(c, \beta) \exp\{-aH(c, \beta) - \gamma_1 a - \gamma_2 c\},$$

where

$$H(c, \beta) = \sum_i \delta_{ui} \Big(\frac{e^{x_i} - 1}{2} \Big)^c + \sum_j \delta_{wj} \Big(\frac{e^{\beta y_j} - 1}{2} \Big)^c + c_u \Big(\frac{e^{\tau} - 1}{2} \Big)^c + c_w \Big(\frac{e^{\beta \tau} - 1}{2} \Big)^c,$$

$$Q(c, \beta) = \prod_j \Big(\frac{e^{\beta y_j} - 1}{2} \Big)^{\delta_{wj}(c-1)} \exp\left(\beta \sum_j \delta_{wj} y_j \right).$$

Since $\gamma_1 > 0$, integration with respect to a can be performed explicitly:

$$\int_0^\infty a^{n_u + n_w + \alpha_1 - 1} e^{-a(H + \gamma_1)} \, da = \frac{\Gamma(n_u + n_w + \alpha_1)}{(H + \gamma_1)^{n_u + n_w + \alpha_1}},$$

which is finite for every (c, β). Hence it remains to verify integrability over (c, β). For $\beta \to 1^+$, all functions appearing in the kernel are continuous, and β^{-1} is bounded on $(1, 1 + \varepsilon)$. Therefore,

$$\int_1^{1+\varepsilon} \pi(a, c, \beta \mid \text{data}) \, d\beta < \infty.$$

Next consider the tail $\beta \to \infty$. Using $e^{\beta z} - 1 \sim e^{\beta z}$, $z > 0$, we obtain $H(c, \beta) \geq c_w 2^{-c} e^{c\beta\tau}$. Moreover,

$$Q(c, \beta) \leq C(c) \exp\left(c\beta \sum_j \delta_{wj} y_j \right),$$

for some positive constant $C(c)$ independent of β. Thus, the posterior kernel satisfies

$$\pi(a, c, \beta \mid \text{data}) \leq M(c) \beta^{n_w - 1} \exp\big(A\beta - Be^{c\beta\tau}\big),$$

where $A = c \sum_j \delta_{wj} y_j$, $B = a\, c_w\, 2^{-c} > 0$. Since $\lim_{\beta\to\infty}(A\beta - Be^{c\beta\tau}) = -\infty$, the exponential term decreases faster than any polynomial growth. Therefore,

$$\int_1^\infty \beta^{n_w - 1} e^{A\beta - Be^{c\beta\tau}}\, d\beta < \infty.$$

Finally, the factor $c^{n_u + n_w + \alpha_2 - 1} e^{-\gamma_2 c}$ is the kernel of a Gamma distribution, hence

$$\int_0^\infty c^{n_u + n_w + \alpha_2 - 1} e^{-\gamma_2 c}\, dc < \infty.$$

Combining the above results yields $K < \infty$. A similar argument exists for the Type-II censoring scheme. In passing, suppose that $\theta = a, c, \beta$. Under the symmetric square error (SE) loss function the Bayesian estimates for $s = 1, 2$ are

$$\hat{\theta}^s_{BSE} = \mathbb{E}(\theta|\text{data}) = \frac{1}{K} \int_1^\infty \int_0^\infty \int_0^\infty \theta L_s(a, c, \beta|\text{data}) \pi(a, c, \beta) da dc d\beta. \tag{8}$$

Also, under asymmetric LINEX loss function for $\zeta \neq 0$, the Bayesian estimates are

$$\hat{\theta}^s_{BL} = -\frac{1}{\zeta} \log \left[\frac{1}{K} \int_1^\infty \int_0^\infty \int_0^\infty e^{-\zeta\theta} L_s(a, c, \beta|\text{data}) \pi(a, c, \beta) da dc d\beta \right]. \tag{9}$$

The size and sign of ζ represent the degree and orientation of asymmetry, respectively. When $\zeta > 0$ overestimation is more costly than underestimation and vice versa. As ζ approaching to zero, the LINEX loss function behaves approximately like the symmetric SE loss function (Zellner, 1986). Under asymmetric general entropy (GE) loss function, the Bayesian estimates for $s = 1, 2$ are

$$\hat{\theta}^s_{BGE} = \Big(\mathbb{E}(\theta^{-\xi}|data) \Big)^{-\frac{1}{\xi}} = \left[\frac{\int_1^\infty \int_0^\infty \int_0^\infty \theta^{-\xi} L_s(a, c, \beta|\text{data}) \pi(a, c, \beta) da dc d\beta}{K} \right]^{-\frac{1}{\xi}}. \tag{10}$$

5.2 E-Bayesian estimation

According to the E-Bayesian approach, the prior distributions $\Gamma(\alpha_1, \gamma_1), \Gamma(\alpha_2, \gamma_2)$, and $\pi_3(\beta)$ should be decreasing functions of a, c and β, respectively. The derivatives of the priors given in (5) show that they are decreasing functions whenever conditions $0 < \alpha_1, \alpha_2 < 1$ and $\gamma_1, \gamma_2 > 0$ hold (Zellner, 1986). Here, we suppose that the

hyperparameters α_i and γ_i for $i = 1, 2$ are independent with the bivariate PDF and given by

$$\begin{aligned}\pi_1(\alpha_i, \gamma_i) &= \pi(\alpha_i)\pi(\gamma_i) = \frac{2\alpha_i}{c_i}, \quad 0 < \alpha_i < 1, 0 < \gamma_i < c_i, \\ \pi_2(\alpha_i, \gamma_i) &= \pi(\alpha_i)\pi(\gamma_i) = \frac{2\gamma_i}{c_i^2}, \quad 0 < \alpha_i < 1, 0 < \gamma_i < c_i, \quad i = 1, 2, \end{aligned} \tag{11}$$

where c_i is a real number. There are other choices to clarify the effect of prior distributions such as $\frac{3\gamma_i^2}{c_i^3}$, but we will focus on the above priors here. Let "BE" denote the Bayesian estimator under the different loss functions given in Section 5.1. Then, the E-Bayesian estimate of a, c and β under the three loss functions and priors (11) can be derived as follows for $L_1(a, c, \beta)$:

$$\begin{aligned}\hat{a}_{EB}^1 &= \iint \hat{a}_{BE}^1 \pi_j(\alpha_1, \gamma_1) d\alpha_1 d\gamma_1, \\ \hat{c}_{EB}^1 &= \iint \hat{c}_{BE}^1 \pi_j(\alpha_2, \gamma_2) d\alpha_2 d\gamma_2, \\ \hat{\beta}_{EB}^1 &= \hat{\beta}_{BE}^1 \end{aligned} \tag{12}$$

and for $L_2(a, c, \beta)$:

$$\begin{aligned}\hat{a}_{EB}^2 &= \iint \hat{a}_{BE}^2 \pi_j(\alpha_1, \gamma_1) d\alpha_1 d\gamma_1, \\ \hat{c}_{EB}^2 &= \iint \hat{c}_{BE}^2 \pi_j(\alpha_2, \gamma_2) d\alpha_2 d\gamma_2, \\ \hat{\beta}_{EB}^2 &= \hat{\beta}_{BE}^2. \end{aligned} \tag{13}$$

Thus, the E-Bayesian estimate of a parameter is the expected Bayesian estimation of that parameter. As we see from (7)-(10) and (12)-(13), the Bayesian and E-Bayesian estimates of a and c can not be derived in closed form under different loss functions. So, we approximate them by applying the MCMC method and Lindley's approximation. To make the Bayesian estimation procedure transparent, we derive the joint posterior distributions under both Type-I and Type-II censoring schemes by explicitly identifying the contributions of failed and censored observations.

5.2.1 Posterior distribution under Type-I censoring

From (1), the likelihood can be written as $L_1 = L_u^{(F)} L_u^{(C)} L_w^{(F)} L_w^{(C)}$, where

$$L_u^{(F)} = \prod_{i=m+1}^{n} f_X(x_i)^{\delta_{ui}},$$

represents the contribution of the observed failures under normal conditions,

$$L_u^{(C)} = \prod_{i=m+1}^{n} S_X(\tau)^{1-\delta_{ui}},$$

corresponds to the censored observations under normal conditions,

$$L_w^{(F)} = \prod_{j=1}^{m} f_Y(y_j)^{\delta_{wj}},$$

denotes the observed failures under accelerated conditions, and

$$L_w^{(C)} = \prod_{j=1}^{m} S_Y(\tau)^{1-\delta_{wj}},$$

is the contribution of the censored observations under accelerated conditions. Substituting the density and survival functions of the WHLD into the above expression yields

$$\begin{aligned} L_1(a,c,\beta) \;\propto\; & a^{n_u+n_w} c^{n_u+n_w} \beta^{n_w} \\ & \times \prod_{i=m+1}^{n} \left(\frac{e^{x_i}-1}{2}\right)^{\delta_{ui}(c-1)} e^{\delta_{ui}x_i} \times \prod_{j=1}^{m} \left(\frac{e^{\beta y_j}-1}{2}\right)^{\delta_{wj}(c-1)} e^{\delta_{wj}\beta y_j} \\ & \times \exp\{-aH(c,\beta)\}. \end{aligned}$$

Multiplying the likelihood by the joint prior gives the posterior kernel

$$\begin{aligned} \pi(a,c,\beta \mid \text{data}) \;\propto\; & a^{n_u+n_w+\alpha_1-1} c^{n_u+n_w+\alpha_2-1} \beta^{n_w-1} \\ & \times \prod_{i=m+1}^{n} \left(\frac{e^{x_i}-1}{2}\right)^{\delta_{ui}(c-1)} \times \prod_{j=1}^{m} \left(\frac{e^{\beta y_j}-1}{2}\right)^{\delta_{wj}(c-1)} \\ & \times \exp\left\{\sum_{j=1}^{m} \delta_{wj}\beta y_j - aH(c,\beta) - \gamma_1 a - \gamma_2 c\right\}. \end{aligned}$$

Therefore, the full conditional distribution of a is obtained analytically as

$$a \mid c, \beta, \text{data} \sim \Gamma\left(n_u + n_w + \alpha_1,\; H(c,\beta) + \gamma_1\right),$$

while the conditional posterior kernels of c and β become

$$\begin{aligned} \pi(c \mid a,\beta,\text{data}) \;\propto\; & c^{n_u+n_w+\alpha_2-1} \prod_{i=m+1}^{n} \left(\frac{e^{x_i}-1}{2}\right)^{\delta_{ui}(c-1)} \\ & \times \prod_{j=1}^{m} \left(\frac{e^{\beta y_j}-1}{2}\right)^{\delta_{wj}(c-1)} \exp\{-aH(c,\beta) - \gamma_2 c\}, \\ \pi(\beta \mid a,c,\text{data}) \;\propto\; & \beta^{n_w-1} \prod_{j=1}^{m} \left(\frac{e^{\beta y_j}-1}{2}\right)^{\delta_{wj}(c-1)} \\ & \times \exp\left\{\sum_{j=1}^{m} \delta_{wj}\beta y_j - aH(c,\beta)\right\}, \end{aligned}$$

which are sampled using the Metropolis–Hastings algorithm.

5.2.2 Posterior distribution under Type-II censoring

Under Type-II censoring, the likelihood has the same decomposition, except that the censoring time is replaced by the terminal order statistics $x_{(r)}$ and $y_{(r)}$. Hence,

$$L_2(a,c,\beta) \propto a^{2r}c^{2r}\beta^r \exp\{-aG(c,\beta)\},$$

where

$$G(c,\beta) = \sum_{i=1}^{r}\left(\frac{e^{x_{(i)}}-1}{2}\right)^c + c_u\left(\frac{e^{x_{(r)}}-1}{2}\right)^c + \sum_{j=1}^{r}\left(\frac{e^{\beta y_{(j)}}-1}{2}\right)^c + c_w\left(\frac{e^{\beta y_{(r)}}-1}{2}\right)^c.$$

Combining L_2 with the prior yields

$$\begin{aligned}\pi(a,c,\beta \mid \text{data}) \;\propto\;& a^{2r+\alpha_1-1}c^{2r+\alpha_2-1}\beta^{r-1} \\ &\times \prod_{i=1}^{r}\left(\frac{e^{x_{(i)}}-1}{2}\right)^{c-1} \times \prod_{j=1}^{r}\left(\frac{e^{\beta y_{(j)}}-1}{2}\right)^{c-1} \\ &\times \exp\left\{\sum_{j=1}^{r} y_{(j)} - aG(c,\beta) - \gamma_1 a - \gamma_2 c\right\},\end{aligned}$$

and then $a \mid c,\beta,\text{data} \sim \Gamma\left(2r+\alpha_1,\ G(c,\beta)+\gamma_1\right)$. The conditional posterior kernels of c and β are obtained analogously by replacing $H(c,\beta)$ with $G(c,\beta)$ and are sampled using the same Metropolis–Hastings within Gibbs algorithm.

5.3 Bayesian estimation using the MCMC method

5.3.1 The MCMC method for Type-I censoring scheme

By (3) for $s=1$, we get the posterior PDFs of a, c and β as

$$\begin{aligned} a|c,\beta,\text{data} \;&\sim\; \Gamma\Big(n_u+n_w+\alpha_1, H(c,\beta)+\gamma_1\Big), \\ \pi(c|a,\beta,\text{data}) \;&\propto\; c^{n_u+n_w+\alpha_2-1}\prod_{i=m+1}^{n}\left(\frac{e^{x_i}-1}{2}\right)^{\delta_{ui}(c-1)} \\ &\quad\times \prod_{j=1}^{m}\left(\frac{e^{\beta y_j}-1}{2}\right)^{\delta_{wj}(c-1)} \times \exp\{-aH(c,\beta)-\gamma_2 c\}, \\ \pi(\beta|a,c,\text{data}) \;&\propto\; \beta^{n_w-1}\prod_{j=1}^{m}\left(\frac{e^{\beta y_j}-1}{2}\right)^{\delta_{wj}(c-1)} \\ &\quad\times \exp\left\{\sum_{j=1}^{m}\delta_{wj}\beta y_j - aH(c,\beta)\right\}. \end{aligned}$$

It is recognized that the posterior PDFs of c and β are not well known distributions. So, we utilize the Metropolis-Hastings method with normal proposal distribution (NPD)

to generate random samples from them. Therefore, the Gibbs sampling algorithm can be proposed as follows:
1. Compute the maximum likelihood estimates $(\hat{a}_1, \hat{c}_1, \hat{\beta}_1)$.
2. Initialize $(a^{(0)}, c^{(0)}, \beta^{(0)}) = (\hat{a}_1, \hat{c}_1, \hat{\beta}_1)$.
3. Set the total number of iterations T and burn-in period N.
4. For $t = 1, \ldots, T$, repeat the following steps:
(a) Gibbs update of a: Generate $a_{(t)} \sim \Gamma\left(n_u + n_w + \alpha_1,\ H(c_{(t-1)}, \beta_{(t-1)}) + \gamma_1\right)$.
(b) Metropolis–Hastings update of c: Generate the proposal $c^* \sim N(c_{(t-1)}, \sigma_c^2)$. Compute the acceptance probability

$$\alpha_c = \min\left\{1,\ \frac{\pi(c^* \mid a_{(t)}, \beta_{(t-1)}, \text{data})}{\pi(c_{(t-1)} \mid a_{(t)}, \beta_{(t-1)}, \text{data})}\right\}.$$

Generate $u \sim U(0,1)$. If $u \leq \alpha_c$, set $c_{(t)} = c^*$, otherwise set $c_{(t)} = c_{(t-1)}$.
(c) Metropolis–Hastings update of β: Generate the proposal $\beta^* \sim N(\beta_{(t-1)}, \sigma_\beta^2)$. Compute

$$\alpha_\beta = \min\left\{1,\ \frac{\pi(\beta^* \mid a_{(t)}, c_{(t)}, \text{data})}{\pi(\beta_{(t-1)} \mid a_{(t)}, c_{(t)}, \text{data})}\right\}.$$

Generate $u \sim U(0,1)$. If $u \leq \alpha_\beta$, set $\beta_{(t)} = \beta^*$, otherwise set $\beta_{(t)} = \beta_{(t-1)}$.
5. After completing T iterations, discard the first N samples as burn-in.
6. Under the squared error loss function, the Bayes estimator is

$$\hat{\theta}_{BSE} = \frac{1}{T-N}\sum_{t=N+1}^{T}\theta_{(t)}, \qquad \theta = a, c, \beta.$$

7. Under the LINEX loss function,

$$\hat{\theta}_{BL} = -\frac{1}{\zeta}\log\left(\frac{1}{T-N}\sum_{t=N+1}^{T} e^{-\zeta\theta_{(t)}}\right).$$

8. Under the general entropy loss function,

$$\hat{\theta}_{BGE} = \left(\frac{1}{T-N}\sum_{t=N+1}^{T}(\theta_{(t)})^{-\xi}\right)^{-1/\xi}.$$

Moreover, a $100(1-\gamma)\%$ HPD credible interval of a, c and β can be constructed by utilizing the method of Chen and Shao (1999).

5.3.2 The MCMC method for Type-II censoring scheme

By (4) for $i = 2$, we get the posterior PDFs of a, c and β as:

$$a|c, \beta, \text{data} \sim \Gamma\Big(2r + \alpha_1, G(c, \beta) + \gamma_1\Big).$$

The conditional posterior kernel of c is

$$\pi(c \mid a, \beta, \text{data}) \quad \propto \quad c^{2r+\alpha_2-1}\prod_{i=1}^{r}\left(\frac{e^{x_{(i)}}-1}{2}\right)^{c-1}$$

$$\times \prod_{j=1}^{r} \left(\frac{e^{\beta y_{(j)}} - 1}{2} \right)^{c-1} \times \exp\{-aG(c,\beta) - \gamma_2 c\}\,.$$

Similarly, the conditional posterior kernel of β is

$$\pi(\beta \mid a, c, \text{data}) \quad \propto \quad \beta^{\,r-1} \prod_{j=1}^{r} \left(\frac{e^{\beta y_{(j)}} - 1}{2} \right)^{c-1} \times \exp\left\{ \beta \sum_{j=1}^{r} y_{(j)} - aG(c,\beta) \right\}, \qquad \beta > 1.$$

So, the conditional posterior distributions of c and β do not belong to standard parametric families. With the same algorithm presented in Subsection 5.3.1, we can estimate $(\hat{a}^2, \hat{c}^2, \hat{\beta}^2)$ under different loss functions.

5.4 Lindley's approximation

One of the most applicable numerical methods to approximate the Bayes estimate has been introduced by Lindley (1980). This method can be used for the model with three parameters $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3) = (a, c, \beta)$. Suppose that $\rho_i = \frac{\partial \rho}{\partial \theta_i}$ where from (6),

$$\rho = \log \pi(a, c, \beta) = (\alpha_1 - 1)\log a + (\alpha_2 - 1)\log c - \log \beta - (\gamma_1 a + \gamma_2 c) + \text{constant}. \quad (14)$$

Let $U = U(\boldsymbol{\theta})$ be an arbitrary function of parameters and $u_i = \frac{\partial u}{\partial \theta_i}$ and $u_{ij} = \frac{\partial^2 u}{\partial \theta_i \partial \theta_j}$ for $i, j = 1, 2, 3$. Also, assume $I^{(s)}_{ijk} = \frac{\partial^3 \ell_s}{\partial \theta_i \partial \theta_j \partial \theta_k}$ and σ^s_{ij} is the (i,j)th element of the matrix $\mathcal{F}_s^{-1}(\boldsymbol{\theta})$ for $s = 1, 2$. By Lindley's approximation (LA),

$$\mathbb{E}(U(\boldsymbol{\theta})|\text{data}) \approx \hat{U}^s_{LA}(\boldsymbol{\theta}) \quad = \quad U(\boldsymbol{\theta}) + \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}(u_{ij} + 2u_i\rho_j)\sigma^s_{ij} + \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} I^{(s)}_{ijk} u_l \sigma^s_{ij}\sigma^s_{kl}\Big|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}^s}, \qquad s = 1, 2.$$

Under SE loss function, $U(\boldsymbol{\theta}) = a$, $U(\boldsymbol{\theta}) = c$ and $U(\boldsymbol{\theta}) = \beta$. Then

$$\hat{a}^{LA}_{SE} = a + \rho_1\sigma^s_{11} + \rho_2\sigma^s_{12} + \rho_3\sigma^s_{13} + \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} I^{(s)}_{ijk}\sigma^s_{ij}\sigma^s_{k1},$$

$$\hat{c}^{LA}_{SE} = c + \rho_1\sigma^s_{21} + \rho_2\sigma^s_{22} + \rho_3\sigma^s_{23} + \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} I^{(s)}_{ijk}\sigma^s_{ij}\sigma^s_{k2},$$

$$\hat{\beta}^{LA}_{SE} = \beta + \rho_1\sigma^s_{31} + \rho_2\sigma^s_{32} + \rho_3\sigma^s_{33} + \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} I^{(s)}_{ijk}\sigma^s_{ij}\sigma^s_{k3}.$$

Under LINEX loss function, $U(\boldsymbol{\theta}) = e^{-\zeta a}$, $U(\boldsymbol{\theta}) = e^{-\zeta c}$ and $U(\boldsymbol{\theta}) = e^{-\zeta \beta}$. Then

$$\hat{a}^{LA}_{LINEX} = -\frac{1}{\zeta}\log\left(e^{-\zeta a} + \frac{1}{2}\sum_{j=1}^{3}(u_{1j} - 2\zeta e^{-\zeta a}\rho_j)\sigma^s_{1j} - \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} I^{(s)}_{ijk}\zeta e^{-\zeta a}\sigma^s_{ij}\sigma^s_{k1}\right),$$

$$\hat{c}_{LINEX}^{LA} = -\frac{1}{\zeta}\log\left(e^{-\zeta c} + \frac{1}{2}\sum_{j=1}^{3}(u_{2j} - 2\zeta e^{-\zeta c}\rho_j)\sigma_{2j}^{s} - \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} I_{ijk}^{(s)}\zeta e^{-\zeta c}\sigma_{ij}^{s}\sigma_{k2}^{s}\right),$$

$$\hat{\beta}_{LINEX}^{LA} = -\frac{1}{\zeta}\log\left(e^{-\zeta \beta} + \frac{1}{2}\sum_{j=1}^{3}(u_{3j} - 2\zeta e^{-\zeta \beta}\rho_j)\sigma_{3j}^{s} - \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} I_{ijk}^{(s)}\zeta e^{-\zeta \beta}\sigma_{ij}^{s}\sigma_{k3}^{s}\right).$$

Under GE loss function, $U(\boldsymbol{\theta}) = a^{-\xi}$, $U(\boldsymbol{\theta}) = c^{-\xi}$ and $U(\boldsymbol{\theta}) = \beta^{-\xi}$. Then

$$\hat{a}_{GE}^{LA} = \left(a^{-\xi} + \frac{1}{2}\sum_{j=1}^{3}(u_{1j} - 2\xi a^{-\xi-1}\rho_j)\sigma_{1j}^{s} - \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} I_{ijk}^{(s)}\xi a^{-\xi-1}\sigma_{ij}^{s}\sigma_{k1}^{s}\right)^{-\frac{1}{\xi}},$$

$$\hat{c}_{GE}^{LA} = \left(c^{-\xi} + \frac{1}{2}\sum_{j=1}^{3}(u_{2j} - 2\xi c^{-\xi-1}\rho_j)\sigma_{2j}^{s} - \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} I_{ijk}^{(s)}\xi c^{-\xi-1}\sigma_{ij}^{s}\sigma_{k2}^{s}\right)^{-\frac{1}{\xi}},$$

$$\hat{\beta}_{GE}^{LA} = \left(\beta^{-\xi} + \frac{1}{2}\sum_{j=1}^{3}(u_{32j} - 2\xi \beta^{-\xi-1}\rho_j)\sigma_{3j}^{s} - \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} I_{ijk}^{(s)}\xi \beta^{-\xi-1}\sigma_{ij}^{s}\sigma_{k3}^{s}\right)^{-\frac{1}{\xi}}.$$

It is noteworthy that all approximations are evaluated at $(\hat{a}^s, \hat{c}^s, \hat{\beta}^s)$ for $s = 1$ and $s = 2$. The elements of the above estimates are presented in detail in Appendix C.

6 Optimal proportion of sample units allocated to accelerated condition

In the test plans, the same number of test units is usually allocated to each stress which is inefficient for estimating the mean life of design stress (Yang , 1994; Kazemi and Zamani Mehreyan, 2024)) In this section, statistically optimum test plans are developed to decide optimum value π^* to each stress. The method adopted here focuses on the Fisher information matrices $\mathcal{F}_s(\boldsymbol{\theta})$ for $s = 1, 2$. Maximizing the determinant of $\mathcal{F}_s$ is equivalent to minimizing the generalized asymptotic variance (GAV) of the estimates of the model parameters. Exactly Bai and Chung (1992):

$$\text{GAV}_i(\text{estimates}) = \frac{1}{|\mathcal{F}_s(\boldsymbol{\theta})|}, \qquad s = 1, 2.$$

It is obvious that the optimal value π^* can be selected such that the GAV is minimized. This optimal criteria is called the D-optimality criterion. Also, we can use another method to obtain the optimal sample proportion π^* allocated to accelerated condition. The criterion can be expressed as follows:

$$\min \ \text{trace}(\mathcal{F}_s^{-1}(\boldsymbol{\theta})), \qquad s = 1, 2.$$

This optimal criteria is called the A-optimality criterion. Accordingly, the corresponding expected optimum number of items failed at use and accelerated conditions can be obtained, respectively, as follows:

$$n_u^* = n(1 - \pi^*)P_u^{(s)}, \qquad n_w^* = n\pi^* P_w^{(s)}, \qquad s = 1, 2,$$

where for Type-I censoring,

$$P_u^{(1)} = \left[1 - e^{-\hat{a}\left(\frac{e^{\tau}-1}{2}\right)^{\hat{c}}}\right], \quad P_w^{(1)} = \left[1 - e^{-\hat{a}\left(\frac{e^{\hat{\beta}\tau}-1}{2}\right)^{\hat{c}}}\right],$$

and for Type-II censoring,

$$P_u^{(2)} = \left[1 - e^{-\hat{a}\left(\frac{e^{x(r)}-1}{2}\right)^{\hat{c}}}\right], \quad P_w^{(2)} = \left[1 - e^{-\hat{a}\left(\frac{e^{\hat{\beta}y_{(r)}}-1}{2}\right)^{\hat{c}}}\right].$$

7 Simulation study

In this section, we perform a simulation study to evaluate the statistical behavior of the estimators and the accuracy of confidence intervals proposed earlier. The main focus is on examining the mean squared error (MSE), performance of confidence intervals and coverage probabilities bias, and average confidence interval lengths (AL) under different censoring schemes and acceleration conditions.

For Type-I censored data, a random sample of $x_1, x_2, \ldots, x_n$ is generated from the WHLD with the parameters $a = 0.7$ and $c = 2$ for different sample sizes $n = 20, 60, 100, 150, 200$. The total sample is divided into two groups: $m = n\pi$ observations (where $\pi = 0.40$) are randomly selected from the n items and assigned to accelerated conditions, and the remaining $n - m$ items stay under normal use.

The lifetimes under normal conditions, X_i, $i = 1, \ldots, n - m$, and under accelerated conditions, Y_j, $j = 1, \ldots, m$, are independent and identically distributed random variables from the WHLD. The accelerated lifetimes follow the relation $Y_j = X_j/\beta$, $j = 1, \ldots, m$ with an acceleration factor $\beta = 3$. Each test item is observed until a censoring time $\tau \in 0.8, 1, 1.5$ without changing the test conditions. The simulation results based on 1000 repetitions, include mean of the estimates, bias, mean squared error(MSE) and %97.5 confidence intervals, empirical empirical coverage probability (CP) and confidence interval length (AL) are presented in Tables 1-7.

Consider $\hat{\boldsymbol{\theta}}_j$ is the estimate of $\boldsymbol{\theta}$ obtained in the jth simulation repetition. MSE and bias of $\boldsymbol{\theta} = (a, \beta, c)$ are given by

$$MSE(\hat{\boldsymbol{\theta}}) = \frac{1}{1000}\sum_{j=1}^{1000}(\hat{\theta}_j - \theta_j)^2, \qquad Bias(\hat{\theta}) = \frac{1}{1000}\sum_{j=1}^{1000}(\hat{\theta}_j - \theta_j),$$

respectively. The CP and AL of the confidence interval are calculated as

$$CP = \frac{1}{1000}\sum_{j=1}^{1000} I(LI_j \le \theta_j \le HI_j), \qquad AL = \frac{1}{1000}\sum_{j=1}^{1000}(HI_j - LI_j),$$

respectively, where LI_j and HI_j denote the lower and upper confidence limits in the jth simulation repetition, respectively, and $I(\cdot)$ is the indicator function.

The MLE, Boot-p, and Boot-t estimators, along with the corresponding %97.5 confidence intervals for the parameters of the WHLD based on CSPALT, are summarized in Tables 1-3 for different sample sizes. The results indicate that, as the sample size

increases, the parameter estimates converge toward their true values, accompanied by a reduction in the MSE. Usually, the bootstrap-t estimator demonstrates superior performance in estimating the shape parameter a, whereas the bootstrap percentile estimator exhibits better accuracy for the parameters β and c.

Overall, the simulation results demonstrate that all three estimation procedures improve substantially as the sample size increases. The biases and MSEs generally decrease toward zero, while the confidence intervals become shorter and the empirical coverage probabilities approach the nominal 97.5% level. Among the considered procedures, the Boot-p method generally provides estimates with smaller biases and MSEs and shorter confidence intervals, particularly for moderate and large sample sizes. The Boot-t method performs reasonably well for large samples, but it exhibits relatively larger biases, MSEs, and average interval lengths in several small- and moderate-sample settings. The MLE also shows consistent behavior, with its estimates approaching the true parameter values as the sample size increases. Overall, the results confirm the consistency and improved finite-sample performance of the proposed estimation procedures under the considered CSPALT settings.

The Markov Chain Monte Carlo (MCMC), Bayesian estimates (BSE) and Lindley's approximations for the parameters of the WHLD based on CSPALT are presented in Tables 5-7. The results show that estimators BSE, SE, LINEX and GE have similar performance in estimating parameters, particularly as the sample size increases. Furthermore, the Bayesian estimators have lower MSE than ML and Boot-p and Boot-t estimators.

For Type-II censored data, similarly a random sample of $x_1, x_2, \ldots, x_n$ is generated from the WHLD with the parameters $a = 0.7$, $\beta = 3$, $c = 2$ and $r = 0.25n$. The maximum likelihood (ML), Boot-p, Boot-t estimates, %97.5 confidence interval and the MCMC Bayesian (BSE) and Lindley's estimates for the parameters of WHLD based on CSPALT under different sample size $n = 20, 60, 100, 150, 200$ are given in the Tables 4-8, respectively. They show that the Bayesian estimators have lower MSE than ML and Boot-p and Boot-t estimators. They also indicate that the estimation accuracy improves as the sample size increases. In particular, the absolute biases and MSEs generally decrease with increasing n, indicating improved estimation accuracy and consistency of the competing methods. The Boot-p method provides particularly small biases and MSEs for the acceleration parameter β, with its bias approaching zero as the sample size increases.

The empirical coverage probabilities of the 97.5% confidence intervals generally increase with the sample size. For $n = 20$, the coverage probabilities are somewhat below the nominal level, whereas for larger sample sizes they approach or exceed the nominal coverage level. At $n = 200$, all reported confidence intervals achieve a coverage probability of 1.0000. In addition, the average interval lengths generally decrease as the sample size increases, reflecting the increasing precision of the interval estimates. Overall, the results demonstrate that increasing the sample size improves both point estimation and interval estimation under the considered censoring scheme.

The optimal value of π^* for each stress and the optimal proportion of sample units allocated to accelerated condition for the simulation studies of Type-I and II censors are calculated and reported in Table 9.

Table 1: Simulation results for estimating the WHLD parameters under CSPALT with $a = 0.7, c = 2, \beta = 3, \pi = 0.40, \tau = 0.8$.

n	Method	Parameter	Estimation	Bias	LI	HI	CP	AL	MSE
20	MLE	a	0.9496	0.2496	0.3443	2.2435	0.9033	1.8992	0.0794
		β	2.8641	-0.1359	1.4834	4.2449	0.9145	2.7615	0.1145
		c	2.2424	0.2424	0.9204	3.5643	0.9174	2.6439	0.0679
	Boot-p	a	0.9395	0.2395	0.6007	1.5936	0.9211	0.9929	0.0793
		β	2.9689	-0.0311	2.2692	4.0320	0.9048	1.7628	0.0862
		c	2.2441	0.2441	1.3472	3.4786	0.9044	2.1314	0.0604
	Boot-t	a	0.9521	0.2521	0.4747	1.3817	0.9044	0.9070	0.0649
		β	3.7644	0.7644	2.3554	4.3692	0.9062	2.0138	0.6121
		c	2.3259	0.3259	0.5690	3.8379	0.9075	3.2689	0.1733
60	MLE	a	0.6140	-0.0860	0.0848	1.1433	0.9357	1.0585	0.2431
		β	3.4521	0.4521	2.2157	4.6886	0.9318	2.4729	0.2639
		c	1.9430	-0.0570	1.2171	2.6689	0.9358	1.4518	0.1125
	Boot-p	a	0.6652	-0.0348	0.4988	0.9440	0.9441	0.4452	0.2405
		β	3.1513	0.1513	2.6102	3.8380	0.9401	1.2278	0.2448
		c	2.1322	0.1322	1.5475	2.9260	0.9486	1.3785	0.1162
	Boot-t	a	0.8676	0.1676	0.5498	0.9422	0.9294	0.3924	0.2357
		β	3.4906	0.4906	2.7303	4.8217	0.9335	2.0914	0.3059
		c	2.2430	0.2430	0.0134	3.4262	0.9429	3.4128	0.1228
100	MLE	a	0.6683	-0.0317	0.2267	1.1098	0.9518	0.8831	0.0229
		β	3.1484	0.1484	2.3335	3.9632	0.9700	1.6297	0.0265
		c	1.9620	-0.0380	1.4038	2.5201	0.9633	1.1163	0.0096
	Boot-p	a	0.6705	-0.0295	0.5526	0.8378	0.9447	0.2852	0.0225
		β	3.1734	0.1734	2.7576	3.6655	0.9526	0.9079	0.0452
		c	1.9587	-0.0413	1.5631	2.4438	0.9739	0.8807	0.0087
	Boot-t	a	0.7551	0.0551	0.3197	0.8812	0.9447	0.5615	0.0205
		β	3.1638	0.1638	2.8537	3.7503	0.9545	0.8966	0.0390
		c	2.1022	0.1022	0.5728	2.8679	0.9682	2.2951	0.0125
150	MLE	a	0.6254	-0.0746	0.3834	1.2673	0.9837	0.8839	0.0117
		β	3.0952	0.0952	2.3423	3.3682	0.9920	1.0259	0.0103
		c	2.0765	0.0765	1.7747	2.7783	0.9967	1.0036	0.0076
	Boot-p	a	0.6848	-0.0152	0.6328	0.9157	0.9949	0.2829	0.0127
		β	3.0000	0.0000	2.6813	3.3694	0.9889	0.6881	0.0117
		c	2.0593	0.0593	1.7268	2.5103	0.9901	0.7835	0.0057
	Boot-t	a	0.6797	-0.0203	0.3885	0.8656	0.9739	0.4771	0.0198
		β	3.1190	0.1190	2.7226	3.5165	0.9765	0.7939	0.0157
		c	2.1029	0.1029	1.1143	2.7732	0.9825	1.6589	0.0119
200	MLE	a	0.6894	-0.0106	0.3687	1.0101	1.0000	0.6414	0.0110
		β	3.0546	0.0546	2.5363	3.5729	1.0000	1.0366	0.0100
		c	2.0388	0.0388	1.6391	2.4384	1.0000	0.7993	0.0038
	Boot-p	a	0.6912	-0.0088	0.5977	0.8083	1.0000	0.2106	0.0108
		β	3.0370	0.0370	2.7549	3.3613	1.0000	0.6064	0.0107
		c	2.0361	0.0361	1.7249	2.3886	1.0000	0.6637	0.0042
	Boot-t	a	0.6889	-0.0111	0.4568	0.7708	1.0000	0.3140	0.0098
		β	3.0607	0.0607	2.8270	3.5602	1.0000	0.7332	0.0149
		c	2.0658	0.0658	0.6812	2.8250	1.0000	2.1438	0.0052

Table 2: Simulation results for estimating the WHLD parameters under CSPALT with $a = 0.7, c = 2, \beta = 3, \pi = 0.40, \tau = 1$.

n	Method	Parameter	Estimation	Bias	LI	HI	CP	AL	MSE
20	MLE	a	0.8108	0.1108	0.0541	1.3675	0.9040	1.3134	0.1083
		β	3.2452	0.2452	1.9136	4.1768	0.9103	2.2632	0.2496
		c	2.0788	0.0788	0.9950	3.1627	0.9329	2.1677	0.0492
	Boot-p	a	0.8079	0.1079	0.6287	0.8147	0.9056	0.1860	0.1081
		β	3.1443	0.1443	2.5387	4.0315	0.9173	1.4928	0.2351
		c	2.1808	0.1808	1.3857	2.9620	0.9246	1.5763	0.0382
	Boot-t	a	0.8068	0.1068	0.2339	0.9765	0.9138	0.7426	0.0975
		β	3.7026	0.7026	2.6543	4.2831	0.9158	1.6288	0.4997
		c	2.2749	0.2749	0.4942	3.2184	0.9257	2.7242	0.1015
60	MLE	a	0.6805	-0.0195	0.3101	1.0509	0.9575	0.7408	0.0404
		β	3.2131	0.2131	2.4978	3.9644	0.9604	1.4666	0.0498
		c	2.0170	0.0170	1.3971	2.6170	0.9615	1.2199	0.0419
	Boot-p	a	0.6914	-0.0086	0.6424	0.7683	0.9288	0.1259	0.0394
		β	3.0537	0.0537	2.6631	3.5008	0.9326	0.8377	0.0381
		c	2.1695	0.1695	1.6806	2.7799	0.9427	1.0993	0.0565
	Boot-t	a	0.7782	0.0782	0.1048	0.8966	0.9329	0.7918	0.0368
		β	3.4552	0.4552	2.7510	4.4733	0.9371	1.7223	0.2577
		c	2.2632	0.2632	0.1327	3.4139	0.9442	3.2812	0.0996
100	MLE	a	0.7064	0.0064	0.4095	1.0033	0.9732	0.5938	0.0211
		β	3.0947	0.0947	2.5119	3.5375	0.9703	1.0256	0.0209
		c	2.0107	0.0107	1.5437	2.4776	0.9738	0.9339	0.0085
	Boot-p	a	0.7054	0.0054	0.6678	0.7466	0.9652	0.0788	0.0210
		β	3.0421	0.0421	2.7578	3.3772	0.9679	0.6194	0.0192
		c	2.0582	0.0582	1.6799	2.3934	0.9705	0.7135	0.0079
	Boot-t	a	0.7518	0.0518	0.3620	0.8339	0.9777	0.4719	0.0202
		β	3.2389	0.2389	2.8205	3.5130	0.9754	0.6925	0.0927
		c	2.1747	0.1747	0.5561	2.3544	0.9708	1.7983	0.0421
150	MLE	a	0.7041	0.0041	0.4608	0.9599	0.9944	0.4991	0.0135
		β	2.9429	-0.0571	2.5616	3.3243	0.9906	0.7627	0.0116
		c	2.0106	0.0106	1.7758	2.6050	0.9993	0.8292	0.0069
	Boot-p	a	0.6922	-0.0078	0.6608	0.7295	0.9929	0.0687	0.0138
		β	3.0273	0.0273	2.7891	3.2979	0.9987	0.5088	0.0125
		c	2.0459	0.0459	1.7594	2.3653	0.9935	0.6059	0.0057
	Boot-t	a	0.7943	0.0943	0.4358	0.8198	0.9967	0.3840	0.0121
		β	3.1198	0.1198	2.8459	3.7113	0.9943	0.8654	0.0159
		c	2.0884	0.0884	0.8703	2.6565	0.9954	1.7862	0.0098
200	MLE	a	0.7026	0.0026	0.4604	0.8649	1.0000	0.4045	0.0109
		β	3.0785	0.0785	2.7061	3.4508	1.0000	0.7447	0.0113
		c	2.0105	0.0105	1.6750	2.3461	1.0000	0.6711	0.0039
	Boot-p	a	0.6949	-0.0051	0.6390	0.7019	1.0000	0.0629	0.0109
		β	3.0170	0.0170	2.8493	3.3073	1.0000	0.4580	0.0110
		c	2.0088	0.0088	1.7591	2.2826	1.0000	0.5235	0.0043
	Boot-t	a	0.7058	0.0058	0.4632	0.7849	1.0000	0.3217	0.0104
		β	3.0392	0.0392	2.9051	3.4008	1.0000	0.4957	0.0141
		c	2.0642	0.0642	1.6005	2.8058	1.0000	1.2053	0.0054

Table 3: Simulation results for estimating the WHLD parameters under CSPALT with $a = 0.7, c = 2, \beta = 3, \pi = 0.40, \tau = 1.5$.

n	Method	Parameter	Estimation	Bias	LI	HI	CP	AL	MSE
20	MLE	a	0.7429	0.0429	0.3049	1.1809	0.9029	0.8760	0.0974
		β	2.9942	-0.0058	2.2832	3.7053	0.9224	1.4221	0.1220
		c	2.1184	0.1184	1.3311	2.9058	0.9419	1.5747	0.3485
	Boot-p	a	0.7774	0.0774	0.5626	0.8593	0.9248	0.2967	0.0985
		β	3.0564	0.0564	2.5614	3.7746	0.9218	1.2132	0.0969
		c	2.1196	0.1196	1.6428	2.6037	0.9228	0.9609	0.3438
	Boot-t	a	0.7977	0.0977	0.3323	0.9286	0.9253	0.5963	0.0864
		β	3.4207	0.4207	2.6476	4.0896	0.9250	1.4420	0.1994
		c	2.2071	0.2071	1.1513	3.3935	0.9247	2.2422	0.3721
60	MLE	a	0.6824	-0.0176	0.4394	0.9254	0.9507	0.4860	0.0373
		β	3.1672	0.1672	2.7149	3.6194	0.9518	0.9045	0.0435
		c	2.0848	0.0848	1.5892	2.4803	0.9587	0.8911	0.0311
	Boot-p	a	0.7614	0.0614	0.5848	0.7969	0.9634	0.1521	0.0385
		β	3.0491	0.0491	2.7553	3.4189	0.9651	0.6636	0.0362
		c	2.1092	0.1092	1.8200	2.4630	0.9749	0.6430	0.0341
	Boot-t	a	0.7708	0.0708	0.1055	0.9168	0.9366	0.8113	0.0342
		β	3.2180	0.2180	2.8257	4.5028	0.9386	1.6771	0.0589
		c	2.1437	0.1437	0.8271	3.4466	0.9402	2.6195	0.0343
100	MLE	a	0.7342	0.0342	0.5339	0.9345	0.9706	0.4006	0.0201
		β	2.9667	-0.0333	2.6379	3.2955	0.9716	0.6576	0.0191
		c	2.0724	0.0724	1.7005	2.3842	0.9701	0.6837	0.0084
	Boot-p	a	0.7320	0.0320	0.6749	0.7878	0.9787	0.1129	0.0205
		β	2.9766	-0.0234	2.7503	3.2472	0.9805	0.4969	0.0182
		c	2.0429	0.0429	1.8254	2.2731	0.9926	0.4477	0.0081
	Boot-t	a	0.7909	0.0909	0.2160	0.8818	0.9779	0.6658	0.0189
		β	3.1437	0.1437	2.7973	3.7403	0.9766	0.9430	0.0244
		c	2.0664	0.0664	1.0832	2.8853	0.9739	1.8021	0.0096
150	MLE	a	0.6872	-0.0128	0.5312	0.8431	0.9941	0.3119	0.0136
		β	2.9675	-0.0325	2.7133	3.2177	0.9901	0.5044	0.0114
		c	2.0712	0.0712	1.8569	2.4456	0.9956	0.5887	0.0065
	Boot-p	a	0.7069	0.0069	0.6602	0.7560	0.9928	0.0958	0.0133
		β	2.9905	-0.0095	2.8019	3.2053	0.9928	0.4034	0.0119
		c	2.0332	0.0332	1.8785	2.2531	0.9980	0.3746	0.0058
	Boot-t	a	0.7569	0.0569	0.3406	0.8032	0.9909	0.4626	0.0129
		β	3.1376	0.1376	2.8424	3.2590	0.9927	0.4166	0.0201
		c	2.0412	0.0412	1.1783	2.5269	0.9912	1.3486	0.0074
200	MLE	a	0.6937	-0.0063	0.5583	0.8292	1.0000	0.2709	0.0103
		β	3.0207	0.0207	2.7884	3.2530	1.0000	0.4646	0.0101
		c	2.0545	0.0545	1.8092	2.2998	1.0000	0.4906	0.0040
	Boot-p	a	0.6947	-0.0053	0.6542	0.7369	1.0000	0.0827	0.0103
		β	3.0032	0.0032	2.8364	3.1897	1.0000	0.3533	0.0097
		c	2.0100	0.0100	1.8863	2.2161	1.0000	0.3298	0.0042
	Boot-t	a	0.7081	0.0081	0.4723	0.7266	1.0000	0.2543	0.0099
		β	3.0993	0.0993	2.8778	3.3882	1.0000	0.5104	0.0121
		c	2.0392	0.0392	1.7593	2.4829	1.0000	0.7236	0.0044

Table 4: Simulation results for estimating the WHLD parameters under CSPALT with $a = 0.7, c = 2, \beta = 3, \pi = 0.40, r = 0.25(n)$.

n	Method	Parameter	Estimation	Bias	LI	HI	CP	AL	MSE
20	MLE	a	0.8480	0.1480	0.0307	1.2267	0.9157	1.1960	0.1060
		β	2.8898	-0.1102	1.8578	4.1218	0.9193	2.2640	0.1510
		c	2.1552	0.1552	0.8780	3.4324	0.9247	2.5544	0.0506
	Boot-p	a	0.8233	0.1233	0.3403	1.4545	0.9270	1.1142	0.1061
		β	2.8778	-0.1222	1.9206	4.4507	0.9257	2.5301	0.1519
		c	2.1525	0.1525	1.3773	3.1030	0.9266	1.7257	0.0502
	Boot-t	a	0.8619	0.1619	0.8097	0.8980	0.9230	0.0883	0.1063
		β	3.2218	0.2218	1.9111	4.8448	0.9217	2.9337	0.1554
		c	2.1732	0.1732	1.1441	3.8767	0.9224	2.7326	0.0515
60	MLE	a	0.7887	0.0887	0.3246	1.1127	0.9367	0.7881	0.0395
		β	3.2260	0.2260	2.5293	3.9228	0.9386	1.3935	0.0583
		c	2.1146	0.1146	1.3928	2.8399	0.9413	1.4471	0.0138
	Boot-p	a	0.7830	0.0830	0.4067	1.5078	0.9574	1.1011	0.0390
		β	2.9964	-0.0036	2.3698	3.7349	0.9578	1.3651	0.0312
		c	2.1133	0.1133	1.4804	2.9164	0.9604	1.4360	0.0138
	Boot-t	a	0.7887	0.0887	0.0707	0.9187	0.9558	0.8480	0.0393
		β	3.1435	0.1435	2.2260	4.9086	0.9590	2.6826	0.0493
		c	2.1051	0.1051	0.2658	3.1995	0.9584	2.9337	0.0236
100	MLE	a	0.7417	0.0417	0.4290	1.0544	0.9588	0.6254	0.0211
		β	3.0620	0.0620	2.4944	3.5180	0.9731	1.0236	0.0211
		c	2.0985	0.0985	1.5355	2.6416	0.9744	1.1061	0.0104
	Boot-p	a	0.7336	0.0336	0.4883	1.2195	0.9691	0.7312	0.0198
		β	2.9946	-0.0054	2.5011	3.5522	0.9680	1.0511	0.0187
		c	2.1105	0.1105	1.6222	2.7649	0.9719	1.1427	0.0149
	Boot-t	a	0.7475	0.0475	0.1714	0.8417	0.9705	0.6703	0.0210
		β	3.0614	0.0614	2.8062	3.1657	0.9726	0.3595	0.0215
		c	2.1019	0.1019	0.4512	2.2627	0.9780	1.8115	0.0129
150	MLE	a	0.7538	0.0538	0.4947	1.0129	0.9749	0.5182	0.0126
		β	2.9100	-0.0900	2.5379	3.2820	0.9930	0.7441	0.0112
		c	2.0415	0.0415	1.7633	2.7197	0.9979	0.9564	0.0067
	Boot-p	a	0.7314	0.0314	0.5218	1.0515	0.9931	0.5297	0.0115
		β	3.0013	0.0013	2.5992	3.4453	0.9829	0.8461	0.0111
		c	2.0412	0.0412	1.6734	2.5415	0.9947	0.8681	0.0067
	Boot-t	a	0.7627	0.0627	0.4660	0.8538	0.9998	0.3878	0.0124
		β	2.9181	-0.0819	2.8986	3.0010	0.9992	0.1024	0.0113
		c	2.0611	0.0611	2.0136	2.3343	0.9994	0.3207	0.0068
200	MLE	a	0.6874	-0.0126	0.4867	0.8881	1.0000	0.4014	0.0104
		β	3.0252	0.0252	2.6617	3.3887	1.0000	0.7270	0.0101
		c	2.0386	0.0386	1.6574	2.4198	1.0000	0.7624	0.0039
	Boot-p	a	0.7241	0.0241	0.5383	0.9923	1.0000	0.4540	0.0102
		β	3.0014	0.0014	2.6494	3.3836	1.0000	0.7342	0.0094
		c	2.0538	0.0538	1.7015	2.4687	1.0000	0.7672	0.0041
	Boot-t	a	0.6923	-0.0077	0.3460	0.7874	1.0000	0.4414	0.0104
		β	3.0351	0.0351	2.8252	3.5611	1.0000	0.7359	0.0102
		c	2.0465	0.0465	1.8968	2.5540	1.0000	0.6572	0.0039

Table 5: The Bayesian and Lindley estimates for the parameters of WHLD based on CSPALT with $a = 0.7, c = 2, \beta = 3, \pi = 0.40, \tau = 0.8$.

		$\hat{a}$	$\hat{\beta}$	$\hat{c}$	$MSE_{\hat{a}}$	$MSE_{\hat{\beta}}$	$MSE_{\hat{c}}$
$n = 60$	BSE	0.8503	2.5442	2.3017	0.2411	0.2185	0.2255
	SE	0.8285	2.6989	2.3529	0.3843	0.2349	0.1936
	LINEX	0.8654	2.7652	2.3372	0.2667	0.3673	0.1587
	GE	0.8097	2.8988	2.2571	0.2428	0.4219	0.1256
$n = 100$	BSE	0.7913	2.9419	2.1608	0.0291	0.0301	0.0074
	SE	0.7888	2.9418	2.1555	0.0276	0.0286	0.0066
	LINEX	0.7971	2.9437	2.1591	0.0299	0.0283	0.0069
	GE	0.7907	2.9527	2.1512	0.0298	0.0278	0.0066
$n = 150$	BSE	0.6924	3.0238	2.1008	0.0099	0.0097	0.0020
	SE	0.6921	3.0230	1.9982	0.0096	0.0094	0.0020
	LINEX	0.7015	3.0241	1.9999	0.0092	0.0093	0.0019
	GE	0.6975	3.0304	1.9943	0.0096	0.0093	0.0021
$n = 200$	BSE	0.7042	2.9984	2.0950	0.0023	0.0030	0.0010
	SE	0.7032	2.9990	2.0934	0.0023	0.0029	0.0010
	LINEX	0.7073	2.9996	2.0942	0.0024	0.0029	0.0010
	GE	0.7061	2.9992	2.0925	0.0024	0.0029	0.0010

Table 6: The Bayesian and Lindley estimates for the parameters of WHLD based on CSPALT with $a = 0.7, c = 2, \beta = 3, \pi = 0.40, \tau = 1$.

		$\hat{a}$	$\hat{\beta}$	$\hat{c}$	$MSE_{\hat{a}}$	$MSE_{\hat{\beta}}$	$MSE_{\hat{c}}$
$n = 20$	BSE	0.8673	2.6748	2.3075	0.0455	0.2368	0.1122
	SE	0.8341	2.6890	2.2293	0.0337	0.2245	0.1164
	LINEX	0.9979	2.7023	2.2693	0.0951	0.2280	0.1689
	GE	0.9526	2.7787	2.2466	0.0748	0.2665	0.1644
$n = 60$	BSE	0.7873	2.8906	2.1942	0.0320	0.0438	0.0526
	SE	0.7863	2.8887	2.1903	0.0313	0.0434	0.0503
	LINEX	0.8051	2.8897	2.1969	0.0343	0.0433	0.0542
	GE	0.7981	2.8963	2.1872	0.0336	0.0427	0.0495
$n = 100$	BSE	0.7963	2.9198	2.0965	0.0204	0.0384	0.0150
	SE	0.7964	2.9180	2.0966	0.0201	0.0382	0.0150
	LINEX	0.8024	2.9179	2.0991	0.0211	0.0381	0.0154
	GE	0.8000	2.9180	2.0975	0.0210	0.0379	0.0153
$n = 150$	BSE	0.6706	3.0396	2.0184	0.0115	0.0122	0.0032
	SE	0.6710	3.0380	2.0186	0.0114	0.0117	0.0032
	LINEX	0.6740	3.0377	2.0198	0.0112	0.0115	0.0032
	GE	0.6725	3.0359	2.0186	0.0114	0.0103	0.0032
$n = 200$	BSE	0.6978	2.9722	2.0122	0.0029	0.0012	0.0016
	SE	0.6978	2.9720	2.0121	0.0029	0.0012	0.0016
	LINEX	0.6992	2.9720	2.0127	0.0032	0.0012	0.0016
	GE	0.6987	2.9722	2.0124	0.0031	0.0012	0.0016

Table 7: The Bayesian and Lindley estimates for the parameters of WHLD based on CSPALT with $a = 0.7, c = 2, \beta = 3, \pi = 0.40, \tau = 1.5$.

		$\hat{a}$	$\hat{\beta}$	$\hat{c}$	$MSE_{\hat{a}}$	$MSE_{\hat{\beta}}$	$MSE_{\hat{c}}$
$n = 20$	BSE	0.7369	2.7966	2.1809	0.0160	0.1857	0.2504
	SE	0.7363	2.7814	2.1996	0.0147	0.1858	0.2703
	LINEX	0.8055	2.7756	2.2363	0.0218	0.1856	0.3083
	GE	0.7729	2.7549	2.2178	0.0187	0.1820	0.2952
$n = 60$	BSE	0.6890	3.0980	2.1558	0.0102	0.0358	0.0291
	SE	0.6892	3.0953	2.1580	0.0101	0.0355	0.0299
	LINEX	0.6964	3.0944	2.1621	0.0199	0.0353	0.0311
	GE	0.6927	3.0911	2.1610	0.0101	0.0344	0.0309
$n = 100$	BSE	0.7274	2.9470	2.0315	0.0095	0.0226	0.0029
	SE	0.7274	2.9461	2.0324	0.0095	0.0226	0.0029
	LINEX	0.7298	2.9458	2.0338	0.0098	0.0226	0.0030
	GE	0.7287	2.9450	2.0338	0.0097	0.0226	0.0031
$n = 150$	BSE	0.6999	3.0541	1.9666	0.0053	0.0133	0.0017
	SE	0.6999	3.0535	1.9670	0.0053	0.0132	0.0017
	LINEX	0.6911	3.0533	1.9676	0.0052	0.0132	0.0017
	GE	0.6905	3.0526	1.9675	0.0052	0.0131	0.0017
$n = 200$	BSE	0.7078	2.9722	2.0252	0.0029	0.0021	0.0016
	SE	0.7078	2.9720	2.0251	0.0029	0.0021	0.0016
	LINEX	0.7092	2.9720	2.0257	0.0032	0.0021	0.0016
	GE	0.7087	2.9722	2.0254	0.0031	0.0021	0.0016

Table 8: The Bayesian and Lindley estimates for the parameters of WHLD based on CSPALT with $a = 0.7, c = 2, \beta = 3, \pi = 0.40 and\ r = 0.25(n)$.

		$\hat{a}$	$\hat{\beta}$	$\hat{c}$	$MSE_{\hat{a}}$	$MSE_{\hat{\beta}}$	$MSE_{\hat{c}}$
$n = 20$	BSE	0.7500	2.8917	1.8530	0.0183	0.1132	0.04837
	SE	0.7588	2.8307	1.8707	0.0169	0.1277	0.04609
	LINEX	0.9045	2.7737	1.8475	0.0611	0.1615	0.03723
	GE	0.8456	2.6012	1.8293	0.0424	0.2155	0.03886
$n = 60$	BSE	0.7461	3.2215	1.8751	0.0108	0.0512	0.0124
	SE	0.7468	3.2090	1.8773	0.0103	0.0471	0.0121
	LINEX	0.7603	3.1859	1.8947	0.0150	0.0388	0.0113
	GE	0.7554	3.1322	1.8939	0.0138	0.0213	0.0114
$n = 100$	BSE	0.7190	3.0646	1.8996	0.0089	0.02546	0.0092
	SE	0.7196	3.0596	1.8905	0.0088	0.0173	0.0091
	LINEX	0.7250	3.0541	1.9036	0.0092	0.0186	0.0087
	GE	0.7224	3.0311	1.9030	0.0091	0.0120	0.0088
$n = 150$	BSE	0.7100	3.0100	1.9097	0.0045	0.0111	0.0049
	SE	0.7102	3.0094	1.9098	0.0045	0.0105	0.0048
	LINEX	0.7135	3.0084	1.9116	0.0041	0.0102	0.0048
	GE	0.7118	3.0035	1.9098	0.0043	0.0102	0.0048
$n = 200$	BSE	0.7024	3.01824	1.9827	0.0030	0.0052	0.0036
	SE	0.7022	3.0121	1.98426	0.0030	0.0051	0.0036
	LINEX	0.7045	3.0117	1.9836	0.0031	0.0051	0.0036
	GE	0.77036	3.0107	1.9828	0.0031	0.0047	0.0036

8 Real data analysis

To demonstrate the applicability of the proposed WHLD and the corresponding inference procedure under the CSPALT scheme, a real data set consisting of failure times of light-emitting diodes (LEDs) is considered. These data have been previously analyzed

Table 9: The optimal value of π^* for each stress and the optimal proportion of sample units allocated to accelerated condition.

Type	τ	n	20	60	100	150	200
Type-I	$\tau = 0.8$	π^*	0.51	0.61	0.61	0.61	0.65
		n_u^*	3	4	9	12	16
		n_w^*	10	41	61	91	130
	$\tau = 1$	π^*	0.51	0.49	0.47	0.46	0.45
		n_u^*	3	14	21	31	36
		n_w^*	1 0	29	47	69	90
	$\tau = 1.5$	π^*	0.47	0.61	0.64	0.64	0.58
		n_u^*	5	19	29	44	72
		n_w^*	9	37	64	96	116
Type-II		π^*	0.70	0.70	0.69	0.68	0.60
		n_u^*	4	11	19	27	24
		n_w^*	4	12	26	39	139

in the reliability and partially accelerated life-testing literature and provide failure-time observations under both normal-use and accelerated-stress conditions, see Fathi et al. (2024). In the present analysis, 20 failure times are considered under normal-use conditions and 22 failure times under accelerated-stress conditions. The failure times are measured in units of 1000 hours.

Let $X_1, \ldots, X_{20}$ denote the failure times under normal-use conditions and $Y_1, \ldots, Y_{22}$ denote the failure times under accelerated-stress conditions. Thus, the total number of observations is $n = 42$, with $m = 22$ observations corresponding to the accelerated condition and $n - m = 20$ observations corresponding to the normal-use condition. Hence, the proportion of accelerated units is $\pi = \frac{m}{n} = \frac{22}{42} = 0.5238$. Since all observations are positive, they are directly compatible with the support of the WHLD, $x > 0$, and no preliminary transformation of the failure times is required.

The original LED observations are complete failure-time data. To illustrate the proposed Type-I CSPALT methodology, a common censoring time of $\tau = 1$ (in units of 1000 hours) is imposed on the data. This censoring time is introduced solely for the purpose of illustrating the proposed Type-I censoring procedure and is not claimed to be the censoring time used in the original LED experiment. Under this censoring scheme, an observation not exceeding τ is regarded as an observed failure, whereas an observation exceeding τ is treated as right-censored. Accordingly, for the normal-use group, 12 observations are less than or equal to τ, yielding $d_u = 12$, $c_u = (n-m) - d_u = 20 - 12 = 8$. Similarly, for the accelerated-stress group, 16 observations are less than or equal to τ, yielding $d_w = 16$, $c_w = m - d_w = 22 - 16 = 6$. Therefore, the configuration used for the real-data illustration is $(n, m, \pi, \tau, d_u, d_w, c_u, c_w) = (42, 22, 0.5238, 1, 12, 16, 8, 6)$. We apply the proposed Type-I censoring scheme with the WHLD to these data and estimate the unknown parameters a, c, and the acceleration factor β using the inferential methods described in the preceding sections. The corresponding estimates, goodness-of-fit results based on the Kolmogorov–Smirnov (K–S) test, and mean squared prediction error (MSE_P) are reported in Table 10.

The results show that the estimated acceleration factor $\hat{\beta}$ is greater than one for all considered methods, with values ranging from 1.1184 to 1.3907, which is consistent with the assumed CSPALT condition $\beta > 1$. The estimates of a range from 2.7338

Table 10: Results for MSE_P and K-S test (with p-value) based on given methods.

Method	$\hat{a}$	$\hat{\beta}$	$\hat{c}$	K-S	p-value	MSE_P
MLE	4.0253	1.1184	1.6487	0.1700	0.9231	0.0805
Boot-p	3.0534	1.2448	1.5090	0.2000	0.8320	0.0946
Boot-t	5.0222	1.3907	1.8874	0.2000	0.8320	0.0736
BSE	2.8009	1.3842	3.4626	0.1500	0.9830	0.0018
SE	2.7338	1.3897	3.4552	0.1600	0.9480	0.0026
LINEX	2.7574	1.3893	3.4630	0.1700	0.9320	0.0023

to 5.0222, whereas the estimates of c range from 1.5090 to 3.4630. In all cases, the estimated value of c is greater than one.

The K–S statistics range from 0.1500 to 0.2000, with the smallest value obtained for the BSE method. The corresponding p-values range from 0.8320 to 0.9831, and all are substantially greater than 0.05. Thus, at the 5% significance level, the K–S test does not provide evidence against the adequacy of the fitted WHLD model for any of the considered estimation methods. More substantial differences are observed in the MSE_P values. The reported prediction errors range from 0.0018 to 0.0946. The BSE method yields the smallest MSE_P value of 0.0018, followed by LINEX and SE with values of 0.0023 and 0.0026, respectively. In comparison, the corresponding values for Boot-t, MLE, and Boot-p are 0.0736, 0.0805, and 0.0946, respectively. Hence, according to the reported MSE_P criterion, the BSE, LINEX, and SE procedures produce substantially smaller prediction errors for the considered LED data.

9 Conclusion

In this paper, we compared among the proposed methods during their findings. The study demonstrated how well the suggested techniques work when applying Type-I and Type-II censoring schemes. The study also made clear how important it is to adopt the CSPALT model, as it yields more failed units in a shorter amount of time than items with typical use situations. Thus, strongly it is suggested that this study be used to tasks like industrial product reliability testing.

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Appendix

Appendix A

$$I_{11}^{(1)} = \frac{\partial^2 \ell_1}{\partial a^2} = -\frac{n_u + n_w}{a^2},$$

$$
\begin{aligned}
I_{22}^{(1)} &= \frac{\partial^2 \ell_1}{\partial c^2} = -\frac{n_u + n_w}{c^2} - \sum_{i=m+1}^{n} \delta_{ui} a \left(\frac{e^{x_i}-1}{2}\right)^c \log^2\left(\frac{e^{x_i}-1}{2}\right) \\
&\quad -c_u a \left(\frac{e^{\tau}-1}{2}\right)^c \log^2\left(\frac{e^{\tau}-1}{2}\right) - \sum_{i=1}^{m} \delta_{wi} a \left(\frac{e^{\beta y_i}-1}{2}\right)^c \log^2\left(\frac{e^{\beta y_i}-1}{2}\right) \\
&\quad -c_w a \left(\frac{e^{\beta \tau}-1}{2}\right)^c \log^2\left(\frac{e^{\beta \tau}-1}{2}\right), \\
I_{33}^{(1)} &= \frac{\partial^2 \ell_1}{\partial \beta^2} \\
&= -\frac{n_w}{\beta^2} + \sum_{i=1}^{m} \delta_{wi} \left[(c-1) y_i^2 e^{\beta y_i} - y_i^2 a c 2^{-c} e^{\beta y_i} (e^{\beta y_i}-1)^{c-2} (c e^{\beta y_i} - 1)\right] \\
&\quad -c_w \tau^2 a c 2^{-c} e^{\beta \tau} (e^{\beta \tau}-1)^{c-2} (c e^{\beta \tau} - 1), \\
I_{12}^{(1)} &= I_{21}^{(1)} = \frac{\partial^2 \ell_1}{\partial a \partial c} \\
&= -\sum_{i=m+1}^{n} \delta_{ui} \left(\frac{e^{x_i}-1}{2}\right)^c \log\left(\frac{e^{x_i}-1}{2}\right) - c_u \left(\frac{e^{\tau}-1}{2}\right)^c \log\left(\frac{e^{\tau}-1}{2}\right) \\
&\quad -\sum_{i=1}^{m} \delta_{wi} \left(\frac{e^{\beta y_i}-1}{2}\right)^c \log\left(\frac{e^{\beta y_i}-1}{2}\right) - c_w \left(\frac{e^{\beta \tau}-1}{2}\right)^c \log\left(\frac{e^{\beta \tau}-1}{2}\right), \\
I_{13}^{(1)} &= I_{31}^{(1)} = \frac{\partial^2 \ell_1}{\partial a \partial \beta} = -\sum_{i=1}^{m} \delta_{wi} c 2^{-c} y_i e^{\beta y_i} (e^{\beta y_i}-1)^{c-1} \\
&\quad -c_w c 2^{-c} \tau e^{\beta \tau} (e^{\beta \tau}-1)^{c-1}, \\
I_{23}^{(1)} &= I_{32}^{(1)} = \frac{\partial^2 \ell_1}{\partial c \partial \beta} \\
&= \sum_{i=1}^{m} \delta_{wi} \left[y_i e^{\beta y_i} - a 2^{-c} y_i e^{\beta y_i} (e^{\beta y_i}-1)^{c-1} \left(1 + c \log\left(\frac{e^{\beta y_i}-1}{2}\right)\right)\right] \\
&\quad -c_w a 2^{-c} \tau e^{\beta \tau} (e^{\beta \tau}-1)^{c-1} \left(1 + c \log\left(\frac{e^{\beta \tau}-1}{2}\right)\right).
\end{aligned}
$$

Appendix B

$$
\begin{aligned}
I_{11}^{(2)} &= \frac{\partial^2 \ell_2}{\partial a^2} = -\frac{2r}{a^2}, \\
I_{22}^{(2)} &= \frac{\partial^2 \ell_2}{\partial c^2} = -\frac{2r}{c^2} - \sum_{i=1}^{r} a \left(\frac{e^{x_{(i)}}-1}{2}\right)^c \log^2\left(\frac{e^{x_{(i)}}-1}{2}\right) \\
&\quad -c_u a \left(\frac{e^{x_{(r)}}-1}{2}\right)^c \log^2\left(\frac{e^{x_{(r)}}-1}{2}\right) - \sum_{j=1}^{r} a \left(\frac{e^{\beta y_{(j)}}-1}{2}\right)^c \log^2\left(\frac{e^{\beta y_{(j)}}-1}{2}\right) \\
&\quad -c_w a \left(\frac{e^{\beta y_{(r)}}-1}{2}\right)^c \log^2\left(\frac{e^{\beta y_{(r)}}-1}{2}\right),
\end{aligned}
$$

$$
\begin{aligned}
I_{33}^{(2)} &= \frac{\partial^2 \ell_2}{\partial \beta^2} \\
&= -\frac{r}{\beta^2} + \sum_{j=1}^{r} \left[(c-1) y_{(j)}^2 e^{\beta y_{(j)}} - y_{(j)}^2 ac 2^{-c} e^{\beta y_{(j)}} (e^{\beta y_{(j)}} - 1)^{c-2} (c e^{\beta y_{(j)}} - 1) \right] \\
&\quad -c_w y_{(r)}^2 ac 2^{-c} e^{\beta y_{(r)}} (e^{\beta y_{(r)}} - 1)^{c-2} (c e^{\beta y_{(r)}} - 1), \\
I_{12}^{(2)} &= I_{21}^{(2)} = \frac{\partial^2 \ell_2}{\partial a \partial c} \\
&= -\sum_{i=1}^{r} \left(\frac{e^{x_{(i)}} - 1}{2} \right)^c \log \left(\frac{e^{x_{(i)}} - 1}{2} \right) - c_u \left(\frac{e^{x_{(r)}} - 1}{2} \right)^c \log \left(\frac{e^{x_{(r)}} - 1}{2} \right) \\
&\quad -\sum_{j=1}^{r} \left(\frac{e^{\beta y_{(j)}} - 1}{2} \right)^c \log \left(\frac{e^{\beta y_{(j)}} - 1}{2} \right) - c_w \left(\frac{e^{\beta y_{(r)}} - 1}{2} \right)^c \log \left(\frac{e^{\beta y_{(r)}} - 1}{2} \right), \\
I_{13}^{(2)} &= I_{31}^{(2)} = \frac{\partial^2 \ell_2}{\partial a \partial \beta} \\
&= -\sum_{j=1}^{r} c 2^{-c} y_{(j)} e^{\beta y_{(j)}} (e^{\beta y_{(j)}} - 1)^{c-1} - c_w c 2^{-c} y_{(r)} e^{\beta y_{(r)}} (e^{\beta y_{(r)}} - 1)^{c-1}, \\
I_{23}^{(2)} &= I_{32}^{(2)} = \frac{\partial^2 \ell_2}{\partial c \partial \beta} \\
&= \sum_{j=1}^{r} \left[y_{(j)} e^{\beta y_{(j)}} - a 2^{-c} y_{(j)} e^{\beta y_{(j)}} (e^{\beta y_{(j)}} - 1)^{c-1} \left(1 + c \log \left(\frac{e^{\beta y_{(j)}} - 1}{2} \right) \right) \right] \\
&\quad -c_w a 2^{-c} y_{(r)} e^{\beta y_{(r)}} (e^{\beta y_{(r)}} - 1)^{c-1} \left(1 + c \log \left(\frac{e^{\beta y_{(r)}} - 1}{2} \right) \right).
\end{aligned}
$$

Appendix C

From (14), we have $\rho_1 = \frac{\alpha_1 - 1}{a} - \gamma_1$, $\rho_2 = \frac{\alpha_2 - 1}{c} - \gamma_2$, $\rho_3 = -\frac{1}{\beta}$.
Under LINEX loss function:

$$
\begin{aligned}
&u_{11} = \zeta^2 e^{-\zeta a}, \quad u_{22} = \zeta^2 e^{-\zeta 2}, \quad u_{33} = \zeta^2 e^{-\zeta \beta}, \\
&u_{12} = u_{13} = u_{21} = u_{23} = u_{31} = u_{32} = 0
\end{aligned}
$$

Under GE loss function:

$$
\begin{aligned}
&u_{11} = \xi(\xi+1) a^{-\xi-2}, \quad u_{22} = \xi(\xi+1) c^{-\xi-2}, \quad u_{33} = \xi(\xi+1) \beta^{-\xi-2} \\
&u_{12} = u_{13} = u_{21} = u_{23} = u_{31} = u_{32} = 0
\end{aligned}
$$

Also for $s = 1$,

$$
\begin{aligned}
I_{111}^{(1)} &= \frac{2(n_u + n_w)}{a^3}, \quad I_{112}^{(1)} = I_{121}^{(1)} = I_{211}^{(1)} = I_{113}^{(1)} = I_{131}^{(1)} = I_{311}^{(1)} = 0, \\
I_{222}^{(1)} &= \frac{2(n_u + n_w)}{c^3} - \sum_{i=m+1}^{n} \delta_{ui} a \left(\frac{e^{x_i} - 1}{2} \right)^c \log^3 \left(\frac{e^{x_i} - 1}{2} \right) \\
&\quad -c_u a \left(\frac{e^{\tau} - 1}{2} \right)^c \log^3 \left(\frac{e^{\tau} - 1}{2} \right)
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{i=1}^{m}\delta_{wi}a\left(\frac{e^{\beta y_i}-1}{2}\right)^{c}\log^3\left(\frac{e^{\beta y_i}-1}{2}\right)-c_w a\left(\frac{e^{\beta\tau}-1}{2}\right)^{c}\log^3\left(\frac{e^{\beta\tau}-1}{2}\right),\\
I^{(1)}_{333}=&\frac{2n_w}{\beta^3+\sum_{i=1}^{m}\delta_{wi}}\\
&\times\Big[(c-1)y_i^3e^{\beta y_i}-a2^{-c}cy_i^3e^{\beta y_i}\left(e^{\beta y_i}-1\right)^{c-3}\left(e^{\beta y_i}\left(c\left(ce^{\beta y_i}-3\right)+1\right)+1\right)\Big]\\
&-c_w a2^{-c}c\tau^3e^{\beta\tau}\left(e^{\beta\tau}-1\right)^{c-3}\left(e^{\beta\tau}\left(c\left(ce^{\beta\tau}-3\right)+1\right)+1\right),\\
I^{(1)}_{122}=&I^{(1)}_{212}=I^{(1)}_{221}\\
=&-\sum_{i=m+1}^{n}\delta_{ui}\left(\frac{e^{x_i}-1}{2}\right)^{c}\log^2\left(\frac{e^{x_i}-1}{2}\right)-c_u\left(\frac{e^{\tau}-1}{2}\right)^{c}\log^2\left(\frac{e^{\tau}-1}{2}\right)\\
&-\sum_{i=1}^{m}\delta_{wi}\left(\frac{e^{\beta y_i}-1}{2}\right)^{c}\log^2\left(\frac{e^{\beta y_i}-1}{2}\right)-c_w\left(\frac{e^{\beta\tau}-1}{2}\right)^{c}\log^2\left(\frac{e^{\beta\tau}-1}{2}\right),\\
I^{(1)}_{132}=&I^{(1)}_{123}=I^{(1)}_{213}=I^{(1)}_{231}=I^{(1)}_{312}=I^{(1)}_{321}\\
=&-\sum_{i=1}^{m}\delta_{wi}2^{-c}y_ie^{\beta y_i}(e^{\beta y_i}-1)^{c-1}\left(1+c\log\left(\frac{e^{\beta y_i}-1}{2}\right)\right)\\
&-c_w2^{-c}\tau e^{\beta\tau}(e^{\beta\tau}-1)^{c-1}\left(1+c\log\left(\frac{e^{\beta\tau}-1}{2}\right)\right),\\
I^{(1)}_{223}=&-\sum_{i=1}^{m}\delta_{wi}a2^{-c}y_ie^{\beta y_i}\left(e^{\beta y_i}-1\right)^{c-1}\log\left(\frac{1}{2}\left(e^{\beta y_i}-1\right)\right)\left(c\log\left(\frac{1}{2}\left(e^{\beta y_i}-1\right)\right)+2\right)\\
&-c_wa2^{-c}\tau e^{\beta\tau}\left(e^{\beta\tau}-1\right)^{c-1}\log\left(\frac{1}{2}\left(e^{\beta\tau}-1\right)\right)\left(c\log\left(\frac{1}{2}\left(e^{\beta\tau}-1\right)\right)+2\right)\\
=&I^{(1)}_{232}=I^{(1)}_{322},\\
I^{(1)}_{331}=&I^{(1)}_{313}=I^{(1)}_{133}=-\sum_{i=1}^{m}\delta_{wi}y_i^2c2^{-c}e^{\beta y_i}(e^{\beta y_i}-1)^{c-2}(ce^{\beta y_i}-1)\\
&-c_w\tau^2c2^{-c}e^{\beta\tau}(e^{\beta\tau}-1)^{c-2}(ce^{\beta\tau}-1),\\
I^{(1)}_{332}=&\sum_{j=1}^{m}\delta_{wi}\Bigg[y_i^2e^{\beta y_i}+\frac{a2^{-c}y_i^2(c(c\log 2-2)e^{\beta y_i}}{e^{-\beta y_i}(e^{\beta y_i}-1)^{2-c}}-c(ce^{\beta y_i}-1)\log(e^{\beta y_i}-1)\\
&-c\log 2+1)\Bigg]+\frac{a2^{-c}\tau^2e^{\beta\tau}(e^{\beta\tau}-1)^{c-2}(c(c\log 2-2)e^{\beta\tau}}{e^{-\beta\tau}(e^{\beta\tau}-1)^{2-c}}\\
&-c(ce^{\beta\tau}-1)\log(e^{\beta\tau}-1)-c\log 2+1)=I^{(1)}_{323}=I^{(1)}_{233}.
\end{aligned}
$$

For $s=2$,

$$
\begin{aligned}
&I^{(2)}_{111}=\frac{2r}{a^3},\quad I^{(2)}_{112}=I^{(2)}_{121}=I^{(2)}_{211}=I^{(2)}_{113}=I^{(2)}_{131}=I^{(2)}_{311}=0,\\
&I^{(2)}_{222}=\frac{2r}{c^3}-\sum_{i=1}^{r}a\left(\frac{e^{x_{(i)}}-1}{2}\right)^{c}\log^3\left(\frac{e^{x_{(i)}}-1}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&-c_u a\left(\frac{e^{x_{(r)}}-1}{2}\right)^c \log^3\left(\frac{e^{x_{(r)}}-1}{2}\right)\\
&-\sum_{j=1}^{r} a\left(\frac{e^{\beta y_{(j)}}-1}{2}\right)^c \log^3\left(\frac{e^{\beta y_{(j)}}-1}{2}\right) - c_w a\left(\frac{e^{\beta y_{(r)}}-1}{2}\right)^c \log^3\left(\frac{e^{\beta y_{(r)}}-1}{2}\right),\\
I^{(2)}_{333} =& \sum_{j=1}^{r}\Big[(c-1)y^3_{(j)}e^{\beta y_{(j)}} - a2^{-c}cy^3_{(j)}e^{\beta y_{(j)}}\left(e^{\beta y_{(j)}}-1\right)^{c-3}\\
&\times\left(e^{\beta y_{(j)}}\left(c\left(ce^{\beta y_{(j)}}-3\right)+1\right)+1\right)\Big]\\
&-c_w a2^{-c}cy^3_{(r)}e^{\beta y_{(r)}}\left(e^{\beta y_{(r)}}-1\right)^{c-3}\left(e^{\beta y_{(r)}}\left(c\left(ce^{\beta y_{(r)}}-3\right)+1\right)+1\right)+\frac{2r}{\beta^3},\\
I^{(2)}_{122} =& I^{(2)}_{212} = I^{(2)}_{221}\\
=& -\sum_{i=1}^{r}\left(\frac{e^{x_{(i)}}-1}{2}\right)^c \log^2\left(\frac{e^{x_{(i)}}-1}{2}\right) - c_u\left(\frac{e^{x_{(r)}}-1}{2}\right)^c \log^2\left(\frac{e^{x_{(r)}}-1}{2}\right)\\
&-\sum_{j=1}^{r}\left(\frac{e^{\beta y_{(j)}}-1}{2}\right)^c \log^2\left(\frac{e^{\beta y_{(j)}}-1}{2}\right) - c_w\left(\frac{e^{\beta y_{(r)}}-1}{2}\right)^c \log^2\left(\frac{e^{\beta y_{(r)}}-1}{2}\right),\\
I^{(2)}_{132} =& I^{(2)}_{123} = I^{(2)}_{213} = I^{(2)}_{231} = I^{(2)}_{312} = I^{(2)}_{321}\\
=& -\sum_{j=1}^{r} 2^{-c}y_{(j)}e^{\beta y_{(j)}}(e^{\beta y_{(j)}}-1)^{c-1}\left(1+c\log\left(\frac{e^{\beta y_{(j)}}-1}{2}\right)\right)\\
&-c_w 2^{-c}\tau e^{\beta y_{(r)}}(e^{\beta y_{(r)}}-1)^{c-1}\left(1+c\log\left(\frac{e^{\beta y_{(r)}}-1}{2}\right)\right),\\
I^{(2)}_{223} =& -\sum_{j=1}^{r} a2^{-c}y_{(j)}e^{\beta y_{(j)}}\left(e^{\beta y_{(j)}}-1\right)^{c-1}\log\left(\frac{1}{2}\left(e^{\beta y_{(j)}}-1\right)\right)\\
&\times\left(c\log\left(\frac{1}{2}\left(e^{\beta y_{(j)}}-1\right)\right)+2\right) - c_w a2^{-c}y_{(r)}e^{\beta y_{(r)}}\left(e^{\beta y_{(r)}}-1\right)^{c-1}\\
&\times\log\left(\frac{1}{2}\left(e^{\beta y_{(r)}}-1\right)\right)\left(c\log\left(\frac{1}{2}\left(e^{\beta y_{(r)}}-1\right)\right)+2\right) = I^{(2)}_{232} = I^{(2)}_{322},\\
I^{(2)}_{331} =& I^{(2)}_{313} = I^{(2)}_{133} = -\sum_{j=1}^{r} y^2_{(j)}c2^{-c}e^{\beta y_{(j)}}(e^{\beta y_{(j)}}-1)^{c-2}(ce^{\beta y_{(j)}}-1)\\
&-c_w y^2_{(r)}c2^{-c}e^{\beta y_{(r)}}(e^{\beta y_{(r)}}-1)^{c-2}(ce^{\beta y_{(r)}}-1),\\
I^{(2)}_{332} =& \sum_{j=1}^{r}\Bigg[y^2_{(j)}e^{\beta y_{(j)}} + \frac{a2^{-c}y^2_{(j)}(c(c\log 2-2)e^{\beta y_{(j)}}}{e^{-\beta y_{(j)}}(e^{\beta y_{(j)}}-1)^{2-c}} - c(ce^{\beta y_{(j)}}-1)\log(e^{\beta y_{(j)}}-1)\\
&-c\log 2+1)\Bigg] + \frac{a2^{-c}y^2_{(r)}e^{\beta y_{(r)}}(e^{\beta y_{(r)}}-1)^{c-2}(c(c\log 2-2)e^{\beta y_{(r)}}}{e^{-\beta y_{(r)}}(e^{\beta y_{(r)}}-1)^{2-c}}\\
&-c(ce^{\beta y_{(r)}}-1)\log(e^{\beta y_{(r)}}-1)-c\log 2+1) = I^{(2)}_{323} = I^{(2)}_{233}.
\end{aligned}
$$