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Extremal two-terminal reliability of directed polyomino-chain networks

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Abstract: We study the two-terminal reliability of directed polyomino-chain networks whose successive squares are attached by either linear or kink extensions. The input and output interfaces are connected to the terminals by perfectly reliable links, whereas all remaining links operate independently with the same probability. Each network is encoded by a binary attachment word, with 0 and 1 representing linear and kink attachments, respectively. Starting from the finite-state transfer-matrix formulation of the model, we establish an exact five-dimensional invariant reduction of the boundary-state dynamics. The two reduced transition matrices corresponding to linear and kink extensions differ by a rank-one matrix. Combining this fact with a reversal symmetry of directed polyomino chains, we prove that replacing any linear attachment by a kink attachment strictly increases the two-terminal reliability. More precisely, if two attachment words satisfy the coordinatewise order $\eta \preceq \xi$ and $\eta \neq \xi$, then $R_\eta(r) < R_\xi(r)$ for $0 < r < 1$. Consequently, within the recursively generated two-terminal network family with some fixed number of squares and the specified input/output interfaces, the all-linear chain is the unique minimizer and the all-kink chain is the unique maximizer of two-terminal reliability. The result provides an exact extremal characterization for this explicitly defined family of planar communication networks; no ordering is claimed for arbitrary polyomino-chain networks or different terminal placements.

Keywords: Extremal network; Invariant subspace; Polyomino chain; Rank-one perturbation; Transfer matrix; Two-terminal reliability.
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Uncorrected Proof

1 Introduction

Network reliability studies the ability of a network to maintain a specified service when some of its links are subject to failure. For a graph with two distinguished terminals s and t, the two-terminal reliability is the probability that the operational subgraph contains an s–t path. Although this quantity is fundamental in the design and analysis of communication and wireless sensor networks (Akyildiz et al., 2002), its exact computation is generally difficult because the number of link-failure configurations grows exponentially with the number of edges (Colbourn, 1987; Ball, 1986). For recursively constructed graph families, however, transfer-matrix and finite-state methods often yield exact reliability polynomials without exhaustive enumeration (Tanguy, 2007).

A related direction in reliability theory concerns extremal problems. Given a prescribed class of networks, one may ask which member has the largest or the smallest two-terminal reliability for every value of the link reliability parameter. Such questions are subtle even for general two-terminal graphs: uniformly most reliable graphs do not exist in many classes with fixed numbers of vertices and edges (Bertrand et al., 2018; Xie et al., 2021). Recent work has also considered the construction of locally most reliable two-terminal graphs (Gong and Lin, 2024). These results show that a uniform extremal ordering is a substantial structural property and must be established within the particular graph family under consideration.

Polyomino chains form a natural class of recursively defined planar graphs. They are constructed from edge-adjacent unit squares whose square-adjacency graph is a path. Their local growth structure has made them suitable for the study of topological indices and extremal problems in chemical graph theory (Golomb, 1994; Ali et al., 2016; Deng et al., 2014; Yarahmadi et al., 2012). Classical distance-based and degree-distance indices, including the Wiener index, the degree distance, and Gutman–Schultz type indices, provide part of the background for this line of research (Wiener, 1947; Dobrynin and Kochetova, 1994; Gutman, 1994). In particular, the Gutman index of deterministic polyomino chains was investigated by Azami and Jafari Rad (2025). Random polyomino-chain models have subsequently been considered for degree-based topological indices, their moments, and their limiting behavior (Sigarreta et al., 2022; Sigarreta and Cruz-Suárez, 2024). Recent work has also developed constructive approaches for extremal degree-based indices of polyomino chains (Montes-y-Morales et al., 2026).

A probabilistic network interpretation of directed polyomino chains was introduced by Azami and Jafari Rad (2026). In that framework, the skeleton graph of a directed polyomino chain is regarded as a communication network, its edges are subject to independent failures, and the source and destination are specified through two boundary interfaces. For a fixed attachment word, an exact transfer-matrix representation was derived by recording the connectivity partition of the source and the three active boundary vertices. This representation makes it possible to compute exact reliability polynomials for all attachment words in the specified family.

The purpose of the present paper is to solve an extremal reliability problem within the *particular recursively generated family and terminal convention defined in Section 2.* Each network in this family with n squares is encoded by a binary attachment word

$$\eta = (\eta_3, \eta_4, \ldots, \eta_n) \in \{0,1\}^{n-2},$$

where 0 denotes a linear attachment and 1 denotes a kink attachment. For two words

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of the same length, we use the coordinatewise order

$$\eta \preceq \xi \quad \Longleftrightarrow \quad \eta_j \leq \xi_j \quad \text{for every } j.$$

Our main result proves that this order is strictly reflected by two-terminal reliability. More precisely, for every $0 < r < 1$,

$$\eta \preceq \xi, \qquad \eta \neq \xi,$$

implies

$$R_\eta(r) < R_\xi(r).$$

Consequently, within the family $\mathcal{C}_n$ explicitly defined in Section 2, the all-linear chain is the unique minimizer and the all-kink chain is the unique maximizer of two-terminal reliability. In particular,

$$R_{0^{n-2}}(r) \leq R_\eta(r) \leq R_{1^{n-2}}(r), \qquad 0 < r < 1,$$

for every attachment word $\eta \in \{0,1\}^{n-2}$, and both inequalities are strict unless η is one of the two extremal words.

The proof is based on the exact transfer-matrix formulation. We show that the reachable boundary-state dynamics admit a five-dimensional invariant representation. Within this reduced representation, the difference between the transition matrices corresponding to a kink and a linear attachment has rank one. Combining this rank-one identity with a reversal symmetry of directed polyomino chains yields a strict local comparison: replacing any one linear attachment by a kink attachment strictly increases two-terminal reliability. The global extremal result then follows by successive local replacements.

The remainder of the paper is organized as follows. Section 2 defines the directed polyomino-chain network model and recalls the transfer-matrix representation. Section 3 establishes the exact five-dimensional invariant reduction. Section 4 proves the rank-one comparison identity and the positivity property needed for the main argument. Section 5 combines these facts with reversal symmetry to prove the strict local improvement theorem. Section 6 derives the global coordinatewise monotonicity theorem and the extremal characterization. Section 7 gives small examples and explains the symbolic verification of the algebraic identities. Section 8 concludes the paper.

2 Network model and transfer representation

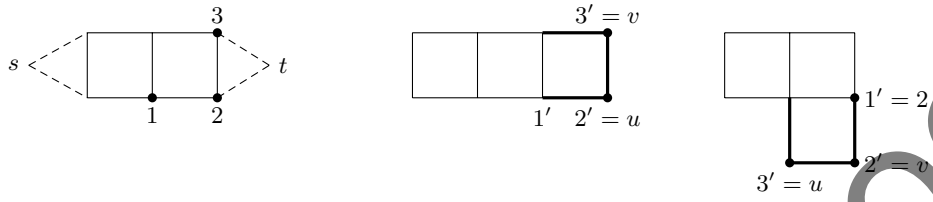
In this section we describe the network model and the transfer representation that will be used in the proof. The word "directed" refers to the prescribed growth direction of the polyomino chain. The reliability event itself is the usual undirected two-terminal connectivity event in the operational subgraph.

We construct a specific polyomino-chain family by starting with two adjacent unit squares and applying the two attachment rules below. After that, each new square is attached in one of two ways. A linear attachment is denoted by 0, and a kink attachment is denoted by 1. Therefore every chain with n squares is encoded by a binary word

$$\eta = (\eta_3, \eta_4, \ldots, \eta_n) \in \{0,1\}^{n-2}.$$

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Here $\eta_k = 0$ means that the k-th square is attached linearly, and $\eta_k = 1$ means that it is attached as a kink.



(a) Two squares (b) Linear, 0 (c) Kink, 1

Figure 1: Exact boundary convention, with unit-square skeleton edges. Dashed edges in (a) are perfectly reliable terminal links (the ordinary grid edges are the only random links). The active triple in (a) is $(1, 2, 3)$; in (b) it is $(2, u, v)$ and in (c) it is $(2, v, u)$. Thick lines indicate newly added ordinary edges. Only active vertices numbered 2 and 3 connect to the output terminal, via perfect links not drawn in (b)–(c).

Explicit geometric initialization. Choose lattice vertices $A = (0, 0)$, $B = (0, 1)$, $C = (1, 0)$, $D = (1, 1)$, $E = (2, 0)$ and $F = (2, 1)$. The initial two-square graph has vertex set $\{A, B, C, D, E, F\}$ and precisely the seven ordinary edges

$$AB,\ CD,\ EF,\ AC,\ CE,\ BD,\ DF.$$

The source s is linked perfectly to A and B. Initially, the active ordered triple is $(1, 2, 3) = (C, E, F)$, where the consecutive edges 12 and 23 are present. For a subsequent step the new square is glued to the existing edge 23 (linear) or 12 (kink); it contributes the three other edges of its four-cycle, in exactly the order specified below. The new active triple is relabelled as given below. After the final step, t is linked perfectly to the vertices labelled 2 and 3. The word "directed" describes the ordered growth process, not the orientation of operational network edges. Figure 1 makes the convention explicit.

Admissible growth and the precise comparison class. Set $e_{\mathrm{E}} = (1, 0)$ and $e_{\mathrm{S}} = (0, -1)$, and let σ interchange e_{E} and e_{S}. For $\omega = (\eta_3, \ldots, \eta_n)$, define the lower-left corners p_k of the successive unit squares by

$$\begin{aligned} &p_1 = (0, 0), \quad p_2 = (1, 0), \quad d_2 = e_{\mathrm{E}}, \\ &d_k = \begin{cases} d_{k-1}, & \eta_k = 0, \\ \sigma(d_{k-1}), & \eta_k = 1, \end{cases} \qquad (3 \le k \le n), \\ &p_k = p_{k-1} + d_k \qquad (3 \le k \le n). \end{aligned}$$

Write $S_k = p_k + [0, 1]^2$ for the corresponding unit squares. After adjoining the perfectly reliable input and output links exactly as above, denote the resulting two-terminal network by $\mathsf{N}_n(\omega)$. The comparison class in this paper is

$$\mathcal{C}_n = \{\mathsf{N}_n(\omega) : \omega \in \{0, 1\}^{n-2}\}.$$

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This explicit definition fixes not only the geometry but also the placement of the terminal interfaces. We make no extremal claim for polyomino chains outside $\mathcal{C}_n$ or for other terminal placements.

Proposition 2.1 (Geometric admissibility of every attachment word)**.** *For every $n \geq 2$ and every $\omega \in \{0,1\}^{n-2}$, the edge-attachment construction above agrees with the lattice construction $(S_k)_{k=1}^{n}$. The squares have pairwise disjoint interiors, and S_i, S_j share a full side if and only if $|i-j| = 1$. In particular, their square-adjacency graph is a path. Each extension introduces exactly two new lattice vertices and three new ordinary edges; thus the resulting network has $3n+1$ ordinary edges.*

Proof. For the second square the active ordered triple $(1,2,3)$ is precisely the south-west, south-east and north-east corners of the current (eastern) square. Abbreviating the compass corners as SW, SE and NE, the active triple is $(\mathrm{SW}, \mathrm{SE}, \mathrm{NE})$ after an eastward step, and $(\mathrm{NE}, \mathrm{SE}, \mathrm{SW})$ after a southward step. These two descriptions are preserved by the prescribed gluing and relabelling rules: in either case, gluing along the edge 23 continues in the same direction (bit 0), whereas gluing along 12 switches east to south or south to east (bit 1). This proves inductively the lattice recurrence for every word.

Each d_k is either $(1,0)$ or $(0,-1)$, so for $j > i$ we have

$$p_j - p_i = \sum_{k=i+1}^{j} d_k = (a,-b), \qquad a,b \in \mathbb{Z}_{\geq 0}, \qquad a+b = j-i.$$

Hence $p_j \neq p_i$ for $j > i$, and their integer unit-square interiors do not overlap. Two such squares share a full side precisely when the Manhattan distance between their lower-left corners is 1. That distance here equals $j-i$, so a shared side occurs exactly for consecutive indices. (Nonconsecutive squares may touch at a corner; such a contact does not create an additional square-adjacency edge.) Finally, an eastward extension places the two new vertices on a strictly new rightmost lattice line; a southward extension places them on a strictly new lowest lattice line. Thus they were not present in any earlier square, and precisely the three nonshared sides of the new square are newly introduced ordinary edges. Starting with seven edges for $n = 2$ gives $7+3(n-2) = 3n+1$ edges, as claimed. □

The input side of the chain is connected to a source vertex s by perfectly reliable auxiliary links. The output side of the chain is connected to a terminal vertex t by perfectly reliable auxiliary links. All other edges are ordinary network links. Each ordinary link is operational with probability r and failed with probability $q = 1-r$, where $0 < r < 1$. The two-terminal reliability is the probability that s is connected to t. It is denoted by $R_\eta(r)$.

To compute $R_\eta(r)$, we keep only the connectivity information on the source s and three active boundary vertices, denoted by

$$1, \quad 2, \quad 3.$$

There are fifteen possible partitions of the set

$$\{s, 1, 2, 3\}.$$

Uncorrected Proof

We use the following fixed order:

$$\begin{array}{lll} S_1 = s|1|2|3, & S_2 = s1|2|3, & S_3 = s2|1|3, \\ S_4 = s3|1|2, & S_5 = s|12|3, & S_6 = s|13|2, \\ S_7 = s|1|23, & S_8 = s1|23, & S_9 = s2|13, \\ S_{10} = s3|12, & S_{11} = s12|3, & S_{12} = s13|2, \\ S_{13} = s23|1, & S_{14} = s|123, & S_{15} = s123. \end{array}$$

Proposition 2.2 (Sufficiency of the fifteen states)**.** *For any already constructed chain, conditioning on the partition of* $\{s, 1, 2, 3\}$ *induced by the operational old edges is sufficient to compute the conditional law of the new partition after either type of attachment and, after any number of further attachments, the final* s*–*t *connection probability.*

Proof. By Proposition 2.1, each extension introduces only the three prescribed new ordinary edges, with no unintended edge identifications. All future ordinary edges attach to the old graph exclusively through its three active vertices; all nonactive old vertices are internal and never receive a new incident edge. Consequently, any path in the augmented operational graph that traverses the old part can be replaced, without changing its endpoints outside that part, by a connection between active vertices or from s to an active vertex. Such connections are recorded exactly by the equivalence classes of the partition of $\{s, 1, 2, 3\}$. Conversely, every equivalence in that partition corresponds to an operational path in the old graph and can be used in any subsequent path. Hence two old edge configurations inducing the same partition give the same new partition for every fixed choice of statuses of the three newly added edges. Since these new statuses are independent of the old configuration, the transition probabilities depend only on the old partition. Iterating proves sufficiency for every extension word. Finally, the perfect output links give success precisely when s belongs to the block containing vertex 2 or 3. □

Independent initialization check. Enumerating the 2^7 operational subsets of the seven initial ordinary edges in Figure 1, joining the source to A and B by perfect links and retaining only the partition of (s, C, E, F) gives precisely the fifteen entries of $\boldsymbol{p}_2$ displayed below. The accompanying independent-enumeration script reproduces each entry as a polynomial in r and $q = 1 - r$.

Let $\boldsymbol{p}_2(r)$, be the initial row vector for the two-square chain. In the above order of states, it is

$$\begin{aligned} \boldsymbol{p}_2(r) \;=\; & (q^5(1+2r), rq^3(1+2rq), 0, r^2q^4, rq^4(1+2r), r^2q^4, rq^4(1+2r), \\ & r^2q^2(1+2rq), 0, r^3q^3, r^2q^2(1+2rq), r^3q^2(3-2r), r^3q^3, \\ & r^2q^3(1+2r) + r^3q^2(2-r), r^3q(1+3rq) + r^4(2-r)(3-2r)). \end{aligned}$$

Now define two 15×15 transition matrices: The matrix $T_0(r)$ corresponds to a linear attachment, and $T_1(r)$ corresponds to a kink attachment.

Suppose that the present active boundary is $(1, 2, 3)$. In a linear attachment, two new vertices u and v are introduced, and the new ordinary edges are

$$(2, u), \qquad (u, v), \qquad (v, 3).$$

The new active boundary is $(2, u, v)$. In a kink attachment, the new ordinary edges are

$$(1, u), \qquad (u, v), \qquad (v, 2),$$

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and the new active boundary is $(2, v, u)$.

For $i = 0, 1$, the entry $(T_i)_{\alpha,\beta}$ is obtained by summing the probabilities of all choices of failed and operational new edges that change state S_α into state S_β. If exactly j of the three new edges are operational, the corresponding probability is $r^j q^{3-j}$. Thus each row of $T_0(r)$ and $T_1(r)$ sums to one.

Constructive algorithm for T_0, T_1. Number the fifteen partitions as above. For each input partition S_α and each bit $i \in \{0, 1\}$, start with the four abstract vertices $s, 1, 2, 3$ identified according to S_α. Introduce u, v, examine each of the eight binary assignments $\epsilon \in \{0, 1\}^3$ to the three edges specified for i, and identify the endpoints of each edge with $\epsilon_j = 1$. Delete all but s and the new ordered active triple, renaming the retained vertices as $(s, 1, 2, 3)$, and find its partition S_β. Add $r^{|\epsilon|} q^{3-|\epsilon|}$ to $(T_i)_{\alpha\beta}$. This uniquely determines every one of the 225 entries of each matrix without relying on an unlisted state convention. Exact code and the complete generated matrices are supplied separately.

Finally, define the success column vector

$$\boldsymbol{c} = (0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 1, 0, 1)^{\mathsf{T}}.$$

In our boundary convention, the two endpoints of the current output interface are represented by the active boundary vertices 2 and 3. Hence a state is successful exactly when the source s is connected to at least one of the vertices 2 and 3. Therefore, for $\eta = (\eta_3, \eta_4, \ldots, \eta_n)$, the two-terminal reliability is

$$R_\eta(r) = \boldsymbol{p}_2(r) T_{\eta_3}(r) T_{\eta_4}(r) \cdots T_{\eta_n}(r) \boldsymbol{c}.$$

For later use, if $\omega = \omega_1 \omega_2 \cdots \omega_m \in \{0, 1\}^m$, we write $T_\omega = T_{\omega_1} T_{\omega_2} \cdots T_{\omega_m}$. For the empty word, we put $T_\emptyset = I_{15}$. Then $R_\omega(r) = \boldsymbol{p}_2(r) T_\omega(r) \boldsymbol{c}$.

The all-linear chain corresponds to the word 0^{n-2}, and the all-kink chain corresponds to the word 1^{n-2}. The main aim of this paper is to prove that the all-linear chain has the smallest two-terminal reliability and the all-kink chain has the largest two-terminal reliability among all chains with the same number of squares.

3 A five-dimensional invariant reduction

In this section we prove that the fifteen-state transfer representation can be reduced to a five-dimensional representation. This reduction is the main algebraic tool of the paper. Throughout the section, let $q = 1 - r$, $0 < r < 1$. If

$$\omega = \omega_1 \omega_2 \cdots \omega_m$$

is a binary word, we write

$$M_\omega = M_{\omega_1} M_{\omega_2} \cdots M_{\omega_m}.$$

For the empty word, we put $M_\emptyset = I_5$.

We define a linear map from five coordinates to the fifteen boundary states. For a row vector (a, b, c, d, e), put

$$\Psi(a, b, c, d, e) \;=\; (a,\ b,\ 0,\ c,\ \frac{r}{q}a,\ d,\ \frac{r}{q}a,\ \frac{r}{q}b,\ 0,\ \frac{r}{q}c,$$

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$$\frac{r}{q}b,\ e,\ \frac{r}{q}c,\ \frac{r^2a+r(2-r)d}{q^2},\ \frac{r^2(b+c)+r(2-r)e}{q^2}).$$

Let Q be the 5×15 matrix defined by

$$(a,b,c,d,e)Q = \Psi(a,b,c,d,e).$$

Now define the initial reduced row vector

$$\boldsymbol{z}_\emptyset = \left(q^5(1+2r), rq^3(1+2rq), r^2q^4, r^2q^4, r^3q^2(3-2r)\right).$$

The reduced transition matrix for a linear attachment is

$$M_0 = \begin{pmatrix} q(1+r) & 0 & 0 & r^2 & 0 \\ q^2(1+r) & rq(1+r) & 0 & r^2q & r^3 \\ q^2 & rq & rq & 0 & r^2 \\ q(1+r-r^2) & 0 & 0 & r^2(2-r) & 0 \\ q^3 & rq(2-r) & rq^2 & 0 & r^2(2-r) \end{pmatrix}.$$

The reduced transition matrix for a kink attachment is

$$M_1 = \begin{pmatrix} q(1+r) & 0 & 0 & r^2 & 0 \\ q^2 & rq & rq & 0 & r^2 \\ q^2(1+r) & rq(1+r) & 0 & r^2q & r^3 \\ q(1+r-r^2) & 0 & 0 & r^2(2-r) & 0 \\ q^3 & rq(2-r) & rq^2 & 0 & r^2(2-r) \end{pmatrix}.$$

Lemma 3.1. *The following identities hold:*

$$\boldsymbol{p}_2 = \boldsymbol{z}_\emptyset Q, \quad QT_0 = M_0 Q, \quad QT_1 = M_1 Q.$$

Proof. The first identity is obtained by multiplying the row vector $\boldsymbol{z}_\emptyset$ by the matrix Q. The result is exactly the initial vector $\boldsymbol{p}_2$ introduced in Section 2.

For the two other identities, we compare the effect of one attachment on the fifteen-state representation and on the five-dimensional representation. A linear attachment gives

$$QT_0 = M_0 Q,$$

and a kink attachment gives

$$QT_1 = M_1 Q.$$

Both identities are checked by direct substitution of the transition rules of Section 2. Equivalently, these are polynomial matrix identities in r and $q = 1 - r$, and they can be verified entry by entry. Hence the row space generated by Q is invariant under the right action of both transfer matrices T_0 and T_1. □

The preceding lemma implies that every reachable fifteen-dimensional row vector has the form of a five-dimensional row vector multiplied by Q.

Lemma 3.2. *For every binary word ω, $\boldsymbol{p}_2 T_\omega = \boldsymbol{z}_\omega Q$, where $\boldsymbol{z}_\omega = \boldsymbol{z}_\emptyset M_\omega$.*

Uncorrected Proof

Proof. We prove the statement by induction on the length of ω. For the empty word, we have $\boldsymbol{p}_2 T_\emptyset = \boldsymbol{p}_2 = \boldsymbol{z}_\emptyset Q$. Thus the statement is true for $\omega = \emptyset$. Assume that $\boldsymbol{p}_2 T_\omega = \boldsymbol{z}_\omega Q$. Let $i \in \{0, 1\}$. Then

$$\boldsymbol{p}_2 T_{\omega i} = \boldsymbol{p}_2 T_\omega T_i = \boldsymbol{z}_\omega Q T_i = \boldsymbol{z}_\omega M_i Q = \boldsymbol{z}_{\omega i} Q.$$

Therefore the result follows for all binary words. □

We now reduce the reliability formula itself. Recall that $R_\omega(r) = \boldsymbol{p}_2 T_\omega \boldsymbol{c}$. By Lemma 3.2, $R_\omega(r) = \boldsymbol{z}_\omega Q \boldsymbol{c}$. A direct multiplication gives $Q\boldsymbol{c} = \boldsymbol{\nu}$, where $\boldsymbol{\nu}' = (0, \frac{r}{q^2}, \frac{1}{q^2}, 0, \frac{1}{q^2})$ Therefore,

$$R_\omega(r) = \boldsymbol{z}_\omega \boldsymbol{\nu}.$$

Thus the reliability of every chain in this family can be computed by the five-dimensional recursion $\boldsymbol{z}_\omega = \boldsymbol{z}_\emptyset M_\omega$, together with the final formula $R_\omega(r) = \boldsymbol{z}_\omega \boldsymbol{\nu}$.

This reduction is exact. No approximation is used. The original fifteen-state transfer process is simply restricted to the five-dimensional invariant row subspace that contains the initial vector $\boldsymbol{p}_2$. The advantage of this reduction is that the difference between the two reduced transition matrices has a simple rank-one form, as shown in the next section.

Remark 3.3 (Scope of the reduction)**.** *The 5×15 matrix Q has full row rank five (columns $1, 2, 4, 6, 12$ contain a 5×5 identity submatrix). The matrix identities $QT_i = M_i Q$ show that $\mathcal{U} = \mathrm{rowspan}(Q)$ is a common invariant five-dimensional row subspace of the full fifteen-state dynamics, independently of the choice of an initial row vector* within *$\mathcal{U}$. The two-square initialization satisfies $\boldsymbol{p}_2 \in \mathcal{U}$ by $\boldsymbol{p}_2 = \boldsymbol{z}_\emptyset Q$. We do* not *claim that every arbitrary initial vector in $\mathbb{R}^{15}$ lies in $\mathcal{U}$ or that this specific five-coordinate formula applies to arbitrary terminal conventions: $\boldsymbol{\nu} = Q\boldsymbol{c}$ uses the output convention of Section 2.*

4 Rank-one comparison and positivity

In this section we compare the two reduced matrices M_0 and M_1. This comparison is the main step toward the proof of the extremal result. Recall that $R_\omega(r) = \boldsymbol{z}_\omega \boldsymbol{\nu}$, where $\boldsymbol{z}_\omega = \boldsymbol{z}_\emptyset M_\omega$, and $\boldsymbol{\nu}' = (0, \frac{r}{q^2}, \frac{1}{q^2}, 0, \frac{1}{q^2})$ For every word ω, write

$$\boldsymbol{z}_\omega = (a_\omega, b_\omega, c_\omega, d_\omega, e_\omega).$$

Define $\boldsymbol{u}' = (0, 1, -1, 0, 0)$, and define the row vector $\boldsymbol{\ell} = rq(-q, -r, 1, -r, r)$.

Lemma 4.1. *We have*

$$M_1 - M_0 = \boldsymbol{u}\boldsymbol{\ell}.$$

In particular, $M_1 - M_0$ has rank one for $0 < r < 1$.

Proof. By subtracting M_0 from M_1, we get

$$M_1 - M_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -rq^2 & -r^2q & rq & -r^2q & r^2q \\ rq^2 & r^2q & -rq & r^2q & -r^2q \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Uncorrected Proof

Also,

$$\boldsymbol{u}\boldsymbol{\ell} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -rq^2 & -r^2q & rq & -r^2q & r^2q \\ rq^2 & r^2q & -rq & r^2q & -r^2q \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore, $M_1 - M_0 = \boldsymbol{u}\boldsymbol{\ell}$. Since both $\boldsymbol{u}$ and $\boldsymbol{\ell}$ are nonzero for $0 < r < 1$, the difference matrix has rank one. This proves the lemma. □

For every word ω, define $\delta_\omega = \boldsymbol{z}_\omega \boldsymbol{u}$. Since $\boldsymbol{u}' = (0, 1, -1, 0, 0)$, we have $\delta_\omega = b_\omega - c_\omega$.

Lemma 4.2. *For every binary word ω, $\delta_\omega > 0$.*

Proof. First we consider the empty word. By the definition of $\boldsymbol{z}_\emptyset$,

$$b_\emptyset = rq^3(1 + 2rq), \quad c_\emptyset = r^2q^4.$$

Thus $\delta_\emptyset = b_\emptyset - c_\emptyset$. Hence $\delta_\emptyset = rq^3(1 + 2rq) - r^2q^4$. Therefore, $\delta_\emptyset = rq^3(1 + rq) > 0$. Now suppose that

$$\boldsymbol{z}_\omega = (a_\omega, b_\omega, c_\omega, d_\omega, e_\omega).$$

A direct multiplication by M_0 gives $\delta_{\omega 0} = rq\big((1 + r)b_\omega + e_\omega\big)$. A direct multiplication by M_1 gives $\delta_{\omega 1} = rq\big((1 + r)c_\omega + e_\omega\big)$.

We also need the fifth coordinate. From the matrices M_0 and M_1, we obtain

$$\begin{aligned} e_{\omega 0} &= r^3 b_\omega + r^2 c_\omega + r^2(2 - r)e_\omega, \\ e_{\omega 1} &= r^2 b_\omega + r^3 c_\omega + r^2(2 - r)e_\omega. \end{aligned}$$

For the empty word, $e_\emptyset = r^3q^2(3 - 2r) > 0$. Moreover, all entries of M_0 and M_1 are nonnegative for $0 < r < 1$. Therefore all coordinates of $\boldsymbol{z}_\omega$ are nonnegative for every word ω. The last two recurrence relations show that if $e_\omega > 0$, then both $e_{\omega 0} > 0$ and $e_{\omega 1} > 0$. Since $e_\emptyset > 0$, it follows by induction that $e_\omega > 0$, for every word ω.

Since $rq > 0$, $1 + r > 0$, and all coordinates of $\boldsymbol{z}_\omega$ are nonnegative, the strict positivity of e_ω implies the strict positivity of both

$$\delta_{\omega 0} = rq\big((1 + r)b_\omega + e_\omega\big), \quad \delta_{\omega 1} = rq\big((1 + r)c_\omega + e_\omega\big).$$

Together with $\delta_\emptyset > 0$, this proves the lemma. □

The conclusion of Lemma 4.2 is important. It says that, for every reachable reduced vector, $b_\omega > c_\omega$. This strict positivity will be used in the next section to prove that changing one attachment from linear to kink strictly increases the two-terminal reliability.

5 Reversal symmetry and strict local improvement

In this section we prove the local comparison theorem. We show that changing one attachment from linear to kink strictly increases the two-terminal reliability.

Let $\omega = \omega_1\omega_2 \ldots \omega_m$, be a binary word. We denote the reversed word by $\omega^R = \omega_m\omega_{m-1} \ldots \omega_1$.

Uncorrected Proof

Lemma 5.1. *For every binary word ω, we have $R_\omega(r) = R_{\omega^R}(r)$.*

Proof. We give an exact algebraic proof from the five-dimensional transfer representation; thus the reversal identity does not depend on an unverified assertion about a drawing or a finite enumeration. Put $q = 1 - r$ and define the following symmetric polynomial matrix:

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -r^3 & rq & 0 & rq(1+r) \\ 0 & rq & -r & 0 & rq \\ 0 & 0 & 0 & 0 & 0 \\ 0 & rq(1+r) & rq & 0 & q(r^2+r-1) \end{pmatrix}.$$

Using the explicit matrices M_0, M_1 from Section 3 and the initial vector $\boldsymbol{z}_\emptyset$, direct polynomial multiplication (with $q = 1 - r$) yields

$$B = B^{\mathsf{T}}, \qquad M_i B = B M_i^{\mathsf{T}} \quad (i = 0, 1), \qquad B\boldsymbol{z}_\emptyset^{\mathsf{T}} = r^2 q^5 \boldsymbol{\nu}.$$

These are identities of explicit polynomial matrices; all entries are checked by the accompanying symbolic-verification program. Notice that B need not be invertible. For every binary word $\omega = \omega_1 \dots \omega_m$, repeated application of the middle identity gives

$$M_\omega B = B M_{\omega^R}^{\mathsf{T}}.$$

Since $r^2 q^5 > 0$ for $0 < r < 1$, and a scalar equals its transpose, we obtain

$$r^2 q^5 R_\omega(r) = \boldsymbol{z}_\emptyset M_\omega B \boldsymbol{z}_\emptyset^{\mathsf{T}} = \boldsymbol{z}_\emptyset B M_{\omega^R}^{\mathsf{T}} \boldsymbol{z}_\emptyset^{\mathsf{T}} = \left(\boldsymbol{z}_\emptyset M_{\omega^R} B \boldsymbol{z}_\emptyset^{\mathsf{T}}\right)^{\mathsf{T}} = r^2 q^5 R_{\omega^R}(r).$$

The third equality uses $B = B^{\mathsf{T}}$. Division by $r^2 q^5$ proves the claimed identity for arbitrary word length. □

For a binary word β, define $\gamma_\beta = \boldsymbol{\ell} M_\beta \boldsymbol{\nu}$. Thus γ_β measures the effect of the suffix β after the rank-one difference $M_1 - M_0$ has been applied. Here $\boldsymbol{\ell}$, M_β, and $\boldsymbol{\nu}$ are the same objects defined in the previous sections.

Lemma 5.2. *For all binary words α and β, we have*

$$R_{\alpha 1 \beta}(r) - R_{\alpha 0 \beta}(r) = \delta_\alpha \gamma_\beta.$$

Proof. Using the five-dimensional formula, we get

$$R_{\alpha 1 \beta}(r) = \boldsymbol{z}_\emptyset M_\alpha M_1 M_\beta \boldsymbol{\nu}, \quad R_{\alpha 0 \beta}(r) = \boldsymbol{z}_\emptyset M_\alpha M_0 M_\beta \boldsymbol{\nu}.$$

Therefore,

$$R_{\alpha 1 \beta}(r) - R_{\alpha 0 \beta}(r) = \boldsymbol{z}_\emptyset M_\alpha (M_1 - M_0) M_\beta \boldsymbol{\nu}.$$

Since $\boldsymbol{z}_\alpha = \boldsymbol{z}_\emptyset M_\alpha$, we have $R_{\alpha 1 \beta}(r) - R_{\alpha 0 \beta}(r) = \boldsymbol{z}_\alpha (M_1 - M_0) M_\beta \boldsymbol{\nu}$. By Lemma 4.1, $M_1 - M_0 = \boldsymbol{u}\boldsymbol{\ell}$. Thus $R_{\alpha 1 \beta}(r) - R_{\alpha 0 \beta}(r) = \boldsymbol{z}_\alpha \boldsymbol{u} \boldsymbol{\ell} M_\beta \boldsymbol{\nu}$. By definition,

$$\boldsymbol{z}_\alpha \boldsymbol{u} = \delta_\alpha, \quad \boldsymbol{\ell} M_\beta \boldsymbol{\nu} = \gamma_\beta.$$

Hence $R_{\alpha 1 \beta}(r) - R_{\alpha 0 \beta}(r) = \delta_\alpha \gamma_\beta$. This proves the lemma. □

Uncorrected Proof

Now, we compute $\gamma_\emptyset$. Since $\boldsymbol{\ell} = rq(-q, -r, 1, -r, r)$ and $\boldsymbol{\nu}' = c(0, r/q^2, 1/q^2, 0, 1/q^2)$, we have

$$\gamma_\emptyset = \boldsymbol{\ell}\boldsymbol{\nu}.$$

Therefore, $\gamma_\emptyset = rq\left(-\frac{r^2}{q^2} + \frac{1}{q^2} + \frac{r}{q^2}\right) = \frac{r(1+r-r^2)}{q}$. Since $1 + r - r^2 = 1 + rq$, we obtain $\gamma_\emptyset = \frac{r(1+rq)}{q}$. On the other hand, from the previous section, $\delta_\emptyset = rq^3(1 + rq)$. Thus $\frac{\gamma_\emptyset}{\delta_\emptyset} = \frac{1}{q^4}$.

Lemma 5.3. *For every binary word β, we have $\gamma_\beta = \frac{\delta_{\beta^R}}{q^4}$.*

Proof. By Lemma 5.2 with $\alpha = \emptyset$, we have $R_{1\beta}(r) - R_{0\beta}(r) = \delta_\emptyset\gamma_\beta$. By Lemma 5.1,

$$R_{1\beta}(r) = R_{\beta^R 1}(r), \quad R_{0\beta}(r) = R_{\beta^R 0}(r).$$

Hence

$$R_{1\beta}(r) - R_{0\beta}(r) = R_{\beta^R 1}(r) - R_{\beta^R 0}(r).$$

Again by Lemma 5.2, with $\alpha = \beta^R$ and $\beta = \emptyset$, we get

$$R_{\beta^R 1}(r) - R_{\beta^R 0}(r) = \delta_{\beta^R}\gamma_\emptyset.$$

Therefore, $\delta_\emptyset\gamma_\beta = \delta_{\beta^R}\gamma_\emptyset$. Since $\delta_\emptyset > 0$, we obtain $\gamma_\beta = \delta_{\beta^R}\frac{\gamma_\emptyset}{\delta_\emptyset}$. Using $\frac{\gamma_\emptyset}{\delta_\emptyset} = \frac{1}{q^4}$, we conclude that $\gamma_\beta = \frac{\delta_{\beta^R}}{q^4}$. This proves the lemma. □

Theorem 5.4. *Let α and β be arbitrary binary words. Then, for every $0 < r < 1$,*

$$R_{\alpha 1\beta}(r) > R_{\alpha 0\beta}(r).$$

Proof. By Lemma 5.2,

$$R_{\alpha 1\beta}(r) - R_{\alpha 0\beta}(r) = \delta_\alpha\gamma_\beta.$$

By Lemma 5.3, $\gamma_\beta = \frac{\delta_{\beta^R}}{q^4}$. Therefore, $R_{\alpha 1\beta}(r) - R_{\alpha 0\beta}(r) = \frac{\delta_\alpha\delta_{\beta^R}}{q^4}$. This identity is the desired exact factorization of the local reliability difference. Since $q > 0$, $\delta_\alpha > 0$, and $\delta_{\beta^R} > 0$, the right-hand side is strictly positive. Thus

$$R_{\alpha 1\beta}(r) - R_{\alpha 0\beta}(r) > 0.$$

Equivalently, $R_{\alpha 1\beta}(r) > R_{\alpha 0\beta}(r)$. This proves the theorem.

□

Theorem 5.4 shows that every single replacement of a linear attachment by a kink attachment strictly increases the two-terminal reliability. This local comparison will be used in the next section to prove the global extremal result.

Proposition 5.5 (Polynomiality and removable denominators)**.** *For each word ω of length m, $R_\omega(r)$ is a polynomial in r of degree at most $7 + 3m$; in particular, the rational terms in the five-coordinate formula have removable denominators. Moreover, for every word ω there is a polynomial $D_\omega(r)$ such that*

$$\delta_\omega = rq^3 D_\omega(r), \qquad q = 1 - r,$$

and the exact local comparison has the manifestly polynomial form

$$R_{\alpha 1\beta}(r) - R_{\alpha 0\beta}(r) = r^2 q^2 D_\alpha(r) D_{\beta^R}(r).$$

Uncorrected Proof

Proof. The seven initial ordinary edges give each entry of $\boldsymbol{p}_2$ as a sum of terms $r^j q^{7-j}$. Each entry of T_i is a sum of terms $r^k q^{3-k}$. The formula $R_\omega = \boldsymbol{p}_2 T_\omega \boldsymbol{c}$ therefore is a polynomial of degree at most $7+3m$ after $q = 1-r$. For the second assertion, each coordinate of $\boldsymbol{z}_\emptyset$ is divisible by q^2 as a polynomial in r, q. Since M_0 and M_1 have polynomial entries, this divisibility persists for every coordinate of $\boldsymbol{z}_\omega$. For the empty word, $\delta_\emptyset = rq^3(1+rq)$. For a nonempty word, the recurrences in Lemma 4.2 read $\delta_{\omega 0} = rq((1+r)b_\omega + e_\omega)$ and $\delta_{\omega 1} = rq((1+r)c_\omega + e_\omega)$. Thus in every case rq^3 divides δ_ω in $\mathbb{Z}[r,q]$, with $q = 1-r$, so D_ω is polynomial. Substitution into Theorem 5.4 gives the stated difference without denominators. This also makes explicit the cancellation of q^{-4} in the factorization. □

6 Global extremal theorem

In this section we derive the global extremal result from the local comparison theorem proved in the previous section.

Let ω and η be two binary words of the same length. We write $\omega \preceq \eta$, if $\omega_j \leq \eta_j$ for every position j. Thus η is obtained from ω by changing some entries from 0 to 1, while no entry is changed from 1 to 0.

Theorem 6.1. *Let ω and η be two binary words of the same length. If*

$$\omega \preceq \eta, \quad \omega \neq \eta,$$

then

$$R_\omega(r) < R_\eta(r), \quad \text{for every } 0 < r < 1.$$

Proof. Since $\omega \leq \eta$ and $\omega \neq \eta$, there is at least one position at which ω has value 0 and η has value 1. Let these positions be changed one by one. In this way we obtain a finite sequence of binary words

$$\omega = \omega^{(0)}, \omega^{(1)}, \ldots, \omega^{(k)} = \eta,$$

where each word is obtained from the preceding one by changing exactly one entry from 0 to 1. For each $j = 0, 1, \ldots, k-1$, the two words $\omega^{(j)}$ and $\omega^{(j+1)}$ have the form

$$\omega^{(j)} = \alpha 0 \beta, \quad \omega^{(j+1)} = \alpha 1 \beta$$

for some binary words α and β. By Theorem 5.4, we have $R_{\omega^{(j)}}(r) < R_{\omega^{(j+1)}}(r)$. Therefore,

$$R_{\omega^{(0)}}(r) < R_{\omega^{(1)}}(r) < \cdots < R_{\omega^{(k)}}(r).$$

Since $\omega^{(0)} = \omega$ and $\omega^{(k)} = \eta$, we conclude that $R_\omega(r) < R_\eta(r)$. This proves the theorem. □

Now let $n \geq 3$. A polyomino chain with n squares is encoded by a binary word of length $n-2$. The all-linear chain corresponds to the word 0^{n-2}, and the all-kink chain corresponds to the word 1^{n-2}.

Uncorrected Proof

Theorem 6.2. *Let $n \geq 3$ and let $\mathcal{C}_n$ be the family of recursively constructed two-terminal networks defined in Section 2. Within $\mathcal{C}_n$, the all-linear word 0^{n-2} uniquely minimizes and the all-kink word 1^{n-2} uniquely maximizes two-terminal reliability for every $0 < r < 1$. More precisely, for every binary word $\omega \in \{0,1\}^{n-2}$,*

$$R_{0^{n-2}}(r) \leq R_\omega(r) \leq R_{1^{n-2}}(r), \qquad 0 < r < 1,$$

and equality on the left holds only when $\omega = 0^{n-2}$, while equality on the right holds only when $\omega = 1^{n-2}$.

Proof. Let ω be any binary word of length $n-2$. Since every entry of ω is either 0 or 1, we have

$$0^{n-2} \preceq \omega \preceq 1^{n-2}.$$

If $\omega \neq 0^{n-2}$, then Theorem 6.1 gives $R_{0^{n-2}}(r) < R_\omega(r)$. Thus the all-linear chain is the unique minimizer. Similarly, if $\omega \neq 1^{n-2}$, then

$$\omega \preceq 1^{n-2}, \quad \omega \neq 1^{n-2}.$$

Hence, by Theorem 6.1, $R_\omega(r) < R_{1^{n-2}}(r)$. Thus the all-kink chain is the unique maximizer. Therefore, $R_{0^{n-2}}(r) \leq R_\omega(r) \leq R_{1^{n-2}}(r)$, and both equalities occur only for the corresponding extremal words. This proves the theorem. □

For $n = 2$, there is no attachment choice, and the family contains only one chain. Therefore the extremal statement is trivial in that case. For every $n \geq 3$, Theorem 6.2 gives the complete extremal characterization of the family.

7 Independent small-case validation

Our small-case check uses direct enumeration of the operational edge subsets of the actual recursively constructed graphs in Figure 1, *not* the transfer matrices. There are $7 + 3(n-2) = 3n+1$ ordinary edges for n squares. For each attachment word ω, and each subset F of those edges, let $I_\omega(F)$ equal one if the source is connected to either endpoint of the final output interface and zero otherwise. Writing $a_{\omega,k} = \sum_{|F|=k} I_\omega(F)$ yields the independently computed exact polynomial

$$R_\omega(r) = \sum_{k=0}^{3n+1} a_{\omega,k} r^k (1-r)^{3n+1-k}.$$

This construction uses only the explicit edge lists and a union–find connectivity test; the 15×15 and 5×5 matrices are not used in deriving the coefficients $a_{\omega,k}$.

Case $n = 3$. For the two possible words, direct enumeration of the 2^{10} edge configurations gives

$$\begin{aligned} R_0(r) &= r^3(2 + 4r - 2r^2 - 13r^3 + 14r^4 - 4r^5), \\ R_1(r) &= r^2(1 + 2r - 9r^4 + 10r^5 - 3r^6). \end{aligned}$$

In particular, an independent subtraction produces

$$R_1(r) - R_0(r) = r^2(1-r)^2(1 + r - r^2)^2 > 0, \qquad (0 < r < 1),$$

Uncorrected Proof

precisely the factorization predicted by Theorem 5.4.

Case $n = 4$. Each of the four words has thirteen ordinary edges. The independently enumerated polynomials are

$$\begin{aligned} R_{00}(r) &= r^4(2 + 6r + 2r^2 - 18r^3 - 15r^4 + 52r^5 - 36r^6 + 8r^7), \\ R_{01}(r) &= r^3(1 + 3r + 2r^2 - 3r^3 - 7r^4 - 11r^5 + 36r^6 - 26r^7 + 6r^8), \\ R_{10}(r) &= R_{01}(r), \\ R_{11}(r) &= r^3(1 + 4r + 3r^2 - 10r^3 - 4r^4 - r^5 + 23r^6 - 20r^7 + 5r^8). \end{aligned}$$

The identity $R_{01} = R_{10}$ therefore follows by direct edge-subset enumeration, independently confirming the first nontrivial reversal identity. In particular,

$$\begin{aligned} R_{01} - R_{00} &= r^3(1 - r)^2(1 + r - r^2)(2r^4 - 4r^3 + 2r + 1), \\ R_{11} - R_{01} &= r^4(1 - r)^2(1 + r - r^2)(r^3 - 3r^2 + 2r + 1). \end{aligned}$$

These two differences coincide exactly with the rank-one factorization after inserting the corresponding prefix/suffix words.

Case $n = 5$ and completeness. For the eight three-bit words, each graph has sixteen ordinary edges. All eight direct edge-enumeration polynomials, including the reversals $R_{001} = R_{100}$ and $R_{011} = R_{110}$, are given in Appendix 8.1. The supplementary CSV records the independent success counts $a_{\omega,k}$ for every $k = 0, \ldots, 16$. For every one of the fourteen words with $n = 3, 4, 5$, we verified the *exact polynomial identity*

$$\sum_k a_{\omega,k} r^k (1 - r)^{3n+1-k} = \boldsymbol{p}_2 T_\omega \boldsymbol{c} = \boldsymbol{z}_\emptyset M_\omega \boldsymbol{\nu}.$$

These are exact symbolic equalities, not approximate numerical checks. In total they compare the outcomes of $2 \cdot 2^{10} + 4 \cdot 2^{13} + 8 \cdot 2^{16} = 559104$ operational-edge configurations. We also generated T_0, T_1 independently from the state-partition procedure and verified both intertwining identities $QT_i = M_i Q$ entry by entry after substituting $q = 1 - r$. The attached programs provide these matrix and direct-enumeration checks in reproducible form.

8 Conclusion

In this paper, we studied the extremal two-terminal reliability problem for directed polyomino-chain networks with independent and identically reliable operational edges. Each network in the family is encoded by a binary word, where a zero represents a linear attachment and a one represents a kink attachment. Starting from the fifteen-state transfer representation, we derived an exact five-dimensional invariant reduction. This reduction allowed us to express the two-terminal reliability in the form $R_\omega(r) = \boldsymbol{z}_\omega \boldsymbol{\nu}$, where $\boldsymbol{z}_\omega = \boldsymbol{z}_\emptyset M_\omega$. The main algebraic observation was the rank-one identity $M_1 - M_0 = \boldsymbol{u}\boldsymbol{\ell}$. Together with the positivity of $\delta_\omega = \boldsymbol{z}_\omega \boldsymbol{u}$, and the reversal symmetry of the network, this identity led to the exact factorization $R_{\alpha 1 \beta}(r) - R_{\alpha 0 \beta}(r) = \frac{\delta_\alpha \delta_{\beta^R}}{q^4}$. Since the right-hand side is strictly positive for $0 < r < 1$, replacing any linear attachment by a kink attachment strictly increases the two-terminal reliability. Consequently, within

Uncorrected Proof

the explicitly defined family $\mathcal{C}_n$ of networks with a fixed number $n \geq 3$ of squares and the specified perfect terminal links, the all-linear chain is the unique minimizer and the all-kink chain is the unique maximizer of two-terminal reliability for every $0 < r < 1$. This is an extremal result for the defined monotone east/south square-gluing construction; it does not assert the same ordering for all conceivable polyomino-chain embeddings or alternative source–terminal interfaces. The results also show that the reliability comparison is not merely a numerical phenomenon. It follows from exact transfer identities, an invariant finite-dimensional reduction, a rank-one comparison, and strict positivity arguments. The accompanying symbolic verification code provides a reproducible check of the main algebraic identities used in the proof.

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Appendix

8.1 Complete $n = 5$ reliability polynomials

The following eight polynomials were obtained by direct enumeration of the 2^{16} subsets of ordinary edges for each word, and then checked against the transfer representation.

$R_{000}(r) = r^5(2 + 8r + 8r^2 - 20r^3 - 44r^4 + 31r^5 + 116r^6 - 172r^7 + 88r^8 - 16r^9)$.
$R_{001}(r) = r^4(1 + 4r + 5r^2 - 3r^3 - 17r^4 - 10r^5 + 9r^6 + 81r^7 - 121r^8 + 64r^9 - 12r^{10})$.
$R_{010}(r) = r^4(1 + 4r + 5r^2 - 4r^3 - 12r^4 - 20r^5 + 19r^6 + 76r^7 - 120r^8 + 64r^9 - 12r^{10})$.
$R_{011}(r) = r^4(1 + 5r + 7r^2 - 9r^3 - 24r^4 + 5r^5 + 24r^6 + 35r^7 - 83r^8 + 50r^9 - 10r^{10})$.
$R_{100}(r) = r^4(1 + 4r + 5r^2 - 3r^3 - 17r^4 - 10r^5 + 9r^6 + 81r^7 - 121r^8 + 64r^9 - 12r^{10})$.
$R_{101}(r) = r^4(2 + 6r + r^2 - 12r^3 - 11r^4 + 14r^5 - 9r^6 + 56r^7 - 83r^8 + 46r^9 - 9r^{10})$.
$R_{110}(r) = r^4(1 + 5r + 7r^2 - 9r^3 - 24r^4 + 5r^5 + 24r^6 + 35r^7 - 83r^8 + 50r^9 - 10r^{10})$.
$R_{111}(r) = r^4(2 + 6r + 2r^2 - 10r^3 - 20r^4 + 12r^5 + 22r^6 + 14r^7 - 57r^8 + 38r^9 - 8r^{10})$.

The equality of reversed pairs 001/100 and 011/110 is visible term by term.

Uncorrected Proof